

C-MOTIVIC EILENBERG-MAC LANE SPECTRA AS MGL-MOTIVIC THOM SPECTRA AND NORMED $\mathbb{G}_m$ -HOCHSCHILD HOMOLOGY

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ABSTRACT. We exhibit the p -complete $\mathbb{C}$ -motivic Eilenberg-Mac Lane spectra MF_p and $(\mathrm{MZ})_p^\wedge$ as $\mathbb{E}_3$ - $\mathrm{MGL}_p^\wedge$ -motivic Thom spectra over the motivic $\mathbb{G}_m$ -loop spaces $\Omega_{\mathbb{G}_m} \mathrm{BGL}$ and $\Omega_{\mathbb{G}_m} \mathrm{BSL}$, respectively. Furthermore, for $p = 2$, we compute the $\mathrm{MGL}_2^\wedge$ -relative $\mathbb{G}_m$ -Hochschild Homology $\mathbb{G}_m \odot^{\mathrm{MGL}_2^\wedge} -$, recently introduced by Bachmann-Burklund-Dundas-Østvær, of MF_2 and $(\mathrm{MZ})_2^\wedge$ to be $\mathrm{MF}_2[\mathrm{BGL}]$ and $(\mathrm{MZ})_2^\wedge[\mathrm{BSL}]$ as $\mathbb{E}_2$ - MF_2 -algebras and $\mathbb{E}_2$ -($\mathrm{MZ})_2^\wedge$ -algebras, respectively.

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1. INTRODUCTION

Motivic homotopy theory is about the emulation of homotopy theory in algebraic geometry. To the author's knowledge, the phenomenon of the Hopkins-Mahowald theorem from homotopy theory hasn't been well emulated in algebraic geometry. The Hopkins-Mahowald theorem, at the prime $p = 2$ is equivalent in math content to each of the four following statements

- (Hopkins-Mahowald Theorem, [Mah79]) There exists a spherical bundle on the topological space $\Omega^2 S^3$ whose Thom spectrum is isomorphic to $\mathbb{F}_2$ as $\mathbb{E}_2$ -rings.
- (Steinberger's calculation, [BMMS86]) The dual Steenrod algebra $\mathbb{F}_2 \otimes \mathbb{F}_2$ is isomorphic to the $\mathbb{F}_2$ homology $\mathbb{F}_2[\Omega^2 S^3]$ of $\Omega^2 S^3$ as $\mathbb{E}_2$ -rings.
- (Bökstedt periodicity, [Bök85]) The topological Hochschild Homology $\mathrm{THH}(\mathbb{F}_2)$ of $\mathbb{F}_2$ is isomorphic to $\mathbb{F}_2[\Omega S^3]$ as $\mathbb{E}_1$ - $\mathbb{F}_2$ -algebras.
- (versal characteristic algebra, [AB19]) There exists a $\mathbb{E}_2$ -ring with universal property of being the free $\mathbb{E}_2$ -ring witnessing a nullhomotopy $2 = 0$, and that $\mathbb{E}_2$ -ring is isomorphic to $\mathbb{F}_2$.

There have been attempts in literature to understand various parts of these equivalent statements in motivic settings. The idea is to try to prove or disprove the following question 1.1 which we think of as asking about the existence of a “motivic Hopkins-Mahowald theorem”.

Question 1.1. *Fix a field k . Does there exist a motivic space $X \in \mathrm{Spc}(k)$ and a motivic spherical bundle on X whose associated motivic Thom spectrum in $\mathrm{SH}(k)$ is isomorphic to the motivic Eilenberg-Mac Lane spectrum MF_2 ?*

In [DHOØ24], Dundas-Hill-Ormsby-Østvær studied a version of (topological) Hochschild homology they call motivic Hochschild Homology. It is constructed by tensoring with the motivic circle $S^1 = S^{1,1}$ in commutative algebras in $\mathrm{SH}(k)$. They then compute motivic Hochschild homology of MF_2 and get a motivic-Hochschild-Homology-of- MF_2 calculation containing τ -torsion.

In [BBDØ26], Bachmann-Burklund-Dundas-Østvær introduce another notion of motivic Hochschild homology $\mathbb{G}_m \odot -$ that differs from the above notion in two ways. Firstly, the tensor is formed in normed motivic spectra [BH21] rather than commutative algebras in motivic spectra. Secondly, the tensor is taken with the motivic space $\mathbb{G}_m = S^{1,0}$ rather than $S^1 = S^{1,1}$. Here the authors compute the motivic Hochschild $\mathbb{G}_m \odot \mathrm{MF}_2$ of MF_2 and show that it is the free $\mathbb{E}_1$ - MF_2 -algebra $\mathrm{MF}_2[\mu_{2,1}]$ on a class $\mu_{2,1}$ in bidegree $(2, 1)$. This computation of motivic Hochschild homology of MF_2 looks more like classical Bökstedt periodicity having no τ -torsion.

Recent work [Cra26] of the author has produced MU-relative versions of the Hopkins-Mahowald theorem and Bökstedt periodicity. These statements seem much more amenable to motivic lifting methods than the do the classical Hopkins-Mahowald theorem and Bökstedt periodicity statement. For example, [Cra26, Theorem A] states that $\mathbb{F}_2$ is isomorphic to the an MU-line bundle on the space U as $\mathbb{E}_3$ -MU-algebras. The motivic lifts of (the classifying space of the base space U) BU and (coefficient ring spectrum of line bundles) MU , motivically refine to the the motivic space BGL and the motivic spectrum MGL . Motivic objects built out of these enjoy niceness properties among being Tate, cellular, and pure. For these reasons, we are able to record a MGL-relative motivic analogue of the Hopkins-Mahowald theorem as Theorem A while we remain unable to answer, positively or negatively, Question 1.1.

Theorem A (Theorem 2). *For every prime p , there exists an $\mathbb{E}_3$ -monoidal p -complete $\mathbb{C}$ -motivic space map*

$$\Omega_{\mathbb{G}_m} BGL \longrightarrow BGL_1(MGL_p^\wedge)$$

whose associated $\mathbb{E}_3$ - $MGL_p^\wedge$ -motivic Thom spectrum is isomorphic to MF_p . Precomposition by the canonical map $\Omega_{\mathbb{G}_m} BSL \rightarrow \Omega_{\mathbb{G}_m} BGL$ yields an $\mathbb{E}_3$ -monoidal p -complete $\mathbb{C}$ -motivic space map

$$\Omega_{\mathbb{G}_m} BSL \longrightarrow BGL_1(MGL_p^\wedge)$$

whose associated $\mathbb{E}_3$ - $MGL_p^\wedge$ -motivic Thom spectrum is isomorphic to $(M\mathbb{Z})_p^\wedge$.

As a corollary we record motivic Thom isomorphisms associated to Theorem A.

Corollary 1.2. *The first motivic Thom spectrum statement of Theorem A has associated Thom isomorphism*

$$\mathrm{MF}_p \otimes_{MGL_p^\wedge} \mathrm{MF}_p \simeq \mathrm{MF}_p[\Omega_{\mathbb{G}_m} BGL]$$

of $\mathbb{E}_3$ - MF_p -algebras. The second motivic Thom spectrum statement of Theorem A has associated Thom isomorphism

$$(M\mathbb{Z})_p^\wedge \otimes_{MGL_p^\wedge} (M\mathbb{Z})_p^\wedge \simeq (M\mathbb{Z})_p^\wedge[\Omega_{\mathbb{G}_m} BSL]$$

of $\mathbb{E}_3$ -($M\mathbb{Z})_p^\wedge$ -algebras.

As a second corollary to Theorem A, we remark on the Hopkins-Hoyois-Morel theorem [Hoy15]. It is the motivic analogue of the Lazard ring calculation. We record the following $\mathbb{E}_3$ -monoidal $\mathbb{C}$ -moivic analogue of the Hopkins-Hoyois-Morel theorem.

Corollary 1.3 (Corollary 2.7). *Let a_i denote the generators of the Lazard ring $(\mathrm{MGL}_{p,*,*}^\wedge)_0 \cong \mathbb{Z}_p^\wedge[a_1, a_2, \dots]$ where $\deg(a_i) = (2i, i)$. Then there exists an equivalence*

$$\mathrm{MGL}_p^\wedge / (a_1, a_2, \dots) \simeq (\mathrm{MZ})_p^\wedge$$

of $\mathbb{E}_3$ - $\mathrm{MGL}_p^\wedge$ -algebras. Furthermore there exists an equivalence

$$\mathrm{MGL}_p^\wedge / (p, a_1, a_2, \dots) \simeq \mathrm{MF}_p$$

of $\mathbb{E}_3$ - $\mathrm{MGL}_p^\wedge$ -algebras.

Remark 1.4. *We note that our proof of Corollary itself depends on the Hopkins-Hoyois-Morel theorem.*

For our other main theorem, Theorem B, for $p = 2$, we record $\mathbb{G}_m \odot -$ associated to the motivic Thom spectra of Theorem A. We warn that $\mathbb{G}_m \odot -$ are defined only for normed motivic spectra and the Thom spectrum statements of theorem A are not stated in $\mathbb{E}_\infty$ form and thus cannot be interpreted in a normed way. Instead, we utilize the corresponding Thom isomorphisms on the level of modules and largely follow the work of [BBDØ26] to prove Theorem B. We make sense of $\mathrm{MGL}_p^\wedge$ -relative $\mathbb{G}_m \odot^{\mathrm{MGL}} -$ by utilizing maps of normed spectra $\mathrm{MGL}_p^\wedge \rightarrow -$ as studied in [BH21].

Theorem B. *There exists an equivalence*

$$\mathbb{G}_m \odot^{\mathrm{MGL}_2^\wedge} \mathrm{MF}_2 \simeq \mathrm{MF}_2[BGL]$$

of $\mathbb{E}_2$ - MF_2 -algebras. Furthermore there exists an equivalence

$$\mathbb{G}_m \odot^{\mathrm{MGL}_2^\wedge} \mathrm{MZ}_2^\wedge \simeq (\mathrm{MZ})_2^\wedge[B\mathbb{S}L]$$

of $\mathbb{E}_2$ -($\mathrm{MZ})_2^\wedge$ -algebras.

Whereas the main theorem of [BBDØ26] shows that $\mathbb{G}_m \odot \mathrm{MF}_2$ is the free $\mathbb{E}_1$ - MF_2 -algebra $\mathrm{MF}_2[\mu_{2,1}]$ on a single generator $\mu_{2,1}$, we conclude the following corollary of Theorem B.

Corollary 1.5. *We have that $\mathbb{G}_m \odot^{\mathrm{MGL}_2^\wedge} \mathrm{MF}_2$ is the product*

$$\mathrm{MF}_2[\mu_{2,1}] \otimes \mathrm{MF}_2[\mu_{4,2}] \otimes \mathrm{MF}_2[\mu_{6,3}] \otimes \dots$$

of free $\mathbb{E}_1$ - MF_2 -algebras $\mathrm{MF}_2[\mu_{2n,n}]$ on classes $\mu_{2n,2}$ of bidegree $(2n, n)$. Likewise $\mathbb{G}_m \odot^{\mathrm{MGL}_2^\wedge} (\mathrm{MZ})_2^\wedge$ is the product

$$\mathrm{MF}_2[\mu_{4,2}] \otimes \mathrm{MF}_2[\mu_{6,3}] \otimes \mathrm{MF}_2[\mu_{8,4}] \otimes \dots$$

of free $\mathbb{E}_1$ - MF_2 -algebras $\mathrm{MF}_2[\mu_{2n,n}]$ on classes $\mu_{2n,2}$ of bidegree $(2n, n)$, starting with $(2n, n) = (4, 2)$.

Remark 1.6. *Our proof of Theorem B, like the proof of the main theorem of [BBDØ26], depends on forthcoming work of Bachmann-Hopkins regarding MF_2 and $(\mathrm{MZ})_2^\wedge$ having normed motivic structure after modding out by τ .*

Conventions. We say category to mean ∞ -category= $(\infty, 1)$ -category. For the remainder of this paper we fix base field $\mathbb{C}$. We refer to the motivic spectra category as $\mathrm{SH}(\mathbb{C})$ and the motivic space category as $\mathrm{Spc}(\mathbb{C})$. We refer to the classifying space construction as B_s to emphasize its simplicial nature. We refer to the $\mathbb{G}_m$ -loop space, S^1 -loop space, and $\mathbb{P}^1$ -loop space as $\Omega_{\mathbb{G}_m}$, Ω_{S^1} , and $\Omega_{\mathbb{P}^1}$, respectively. When we write $\mathbb{F}_p$, we either mean the discrete commutative ring or the Eilenberg-Mac Lane spectrum whereas we (try to) always write MF_p for the motivic Eilenberg-Mac Lane spectrum. In sections 2 and 3 we always p -complete everything, and in section 3, $p = 2$ always. We warn that we use BGL to both denote the motivic space BGL as well as the space $\mathrm{BGL} \subset \mathrm{Pic}$ that is the connected component of Pic containing the identity. We write BSL to denote the motivic space BSL and well as write K to denote the algebraic K-theory motivic space $\mathbb{Z} \times \mathrm{BGL}$. We write $\mathfrak{R}$ to denote Betti realization.

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2. GENERALIZED $\mathbb{C}$ -MOTIVIC THOM SPECTRA

The purpose of this section is to prove Theorem A concerning motivic Thom spectra. We begin by recalling facts.

Recollection 2.1. *The functor*

$$\mathrm{fil}_{\mathrm{ev}} : \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}_{\mathrm{fil}_{\mathrm{ev}}(\mathbb{S})} \mathrm{Fil} \mathrm{Sp}) \rightarrow \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{SH}(\mathbb{C})^{\mathrm{cell}})$$

given by Pstragowski's even filtration [Pst23a] induces an equivalence after p -completion between even MU -synthetic spectra, modeled by modules filtered spectra on $\mathrm{fil}_{\mathrm{ev}}^*(\mathbb{S})$ and cellular motivic spectra.

The following fact is standard.

Lemma 2.2. *Let $\mathcal{C}$ denote a presentably symmetric monoidal category, $n > 0$, and $f : X \rightarrow \mathrm{Pic}(\mathcal{C})$ denote an $\mathbb{E}_n$ -monoidal space map where X is grouplike and connected. Then the Thom object $T(f)$ associated to the colimit of f identifies, as $\mathbb{E}_n$ -algebras in $\mathcal{C}$, with*

$$id_{\mathcal{C}} \otimes_{id_{\mathcal{C}[\Omega X]}} id_{\mathcal{C}}.$$

From [Cra26] we recall there exists an $\mathbb{E}_3$ -space map $U \rightarrow \mathrm{BGL}_1(\mathrm{MU})$ whose $\mathbb{E}_3$ - MU -Thom spectrum is isomorphic to $\mathbb{F}_2$ and furthermore precomposing with $SU \hookrightarrow U$ yields an $\mathbb{E}_3$ - MU -space map $SU \rightarrow \mathrm{BGL}_1(\mathrm{MU})$ whose $\mathbb{E}_3$ - MU -Thom spectrum is isomorphic to $\mathbb{Z}$.

Lemma 2.3. *Suppose X is a motivic space whose motivic suspension spectrum $\Sigma_+^\infty X$ is an even finite MGL-projective motivic spectrum. Then the natural map*

$$\mathrm{MGL}[X] \rightarrow \mathrm{fil}_{\mathrm{ev}}(\mathrm{MU}[\mathrm{Re}(X)])$$

is an equivalence of $\mathbb{E}_\infty$ -MGL-algebras.

Proof. By the finite MGL-projectiveness assumption on $\Sigma_+^\infty X$ we have that the natural map

$$(\mathrm{MGL}_{*,*}[X] \rightarrow \mathrm{MU}_* \mathrm{Re}(X))$$

is an equivalence of graded MU_* -modules by [Pst23b, Lemma 7.9]. The claim follows from evenness. $\square$

Lemma 2.4. *Suppose X is grouplike pointed $\mathbb{E}_n$ -space such whose motivic suspension spectrum $\mathrm{fil}_{\mathrm{ev}} \Sigma_+^\infty \Omega X$ is an even finite MGL-projective motivic spectrum. and $f: X \rightarrow BGL_1(\mathrm{MU})$ is an $\mathbb{E}_n$ -space map whose associated $\mathbb{E}_n$ -MU-Thom spectrum we denote $T(f)$. Then the natural map*

$$\mathrm{fil}_{\mathrm{ev}}(\mathrm{MU}) \otimes_{\mathrm{fil}_{\mathrm{ev}}(\mathrm{MU}[\Omega X])} \mathrm{fil}_{\mathrm{ev}}(\mathrm{MU}) \rightarrow \mathrm{fil}_{\mathrm{ev}}(\mathrm{MU} \otimes_{\mathrm{MU}[\Omega X]} \mathrm{MU})$$

is an equivalence of $\mathbb{E}_n$ - $\mathrm{fil}_{\mathrm{ev}}(\mathrm{MU})$ -algebras.

Proof. This associated trigraded Tor sepctral sequence with signature

$$E_{s,t,w}^2 = \mathrm{Tor}_s^{\pi_{*,*}(\mathrm{fil}_{\mathrm{ev}}(\mathrm{MU}[\Omega X]))}(\pi_{*,*}(\mathrm{fil}_{\mathrm{ev}}(\mathrm{MU})), \pi_{*,*}(\mathrm{fil}_{\mathrm{ev}}(\mathrm{MU})))_{t,w} \Rightarrow \pi_{s+t,w}(\mathrm{fil}_{\mathrm{ev}}(\mathrm{MU} \otimes_{\mathrm{MU}[\Omega X]} \mathrm{MU}))$$

whose convergence is attributed to cellularity. This spectral sequence collapses on the E^2 page for evenness degree reasons and there are no extension problems to solve for filtration reasons. $\square$

Lemma 2.5. *The natural maps motivic spaces*

$$B_s K \rightarrow \Omega_{\mathbb{G}_m} BGL$$

and

$$B_s BGL \rightarrow \Omega_{\mathbb{G}_m} BSL$$

that are equivalences of motivic spaces.

Proof. By motivic Bott periodicity [GS09, Proposition 4.1], the natural map

$$K \rightarrow \Omega_{\mathbb{P}^1} K$$

is an equivalence. By observing $\Omega_{\mathbb{P}^1} K = \Omega_s \Omega_{\mathbb{G}_m} K$ and applying B_s , one obtains the equivalence

$$B_s K \rightarrow B_s \Omega_s \Omega_{\mathbb{G}_m} K$$

from which one concludes the equivalence

$$B_s K \rightarrow \Omega_{\mathbb{G}_m} K.$$

The desired first statement follows from considering the fiber sequence

$$BGL \rightarrow K \rightarrow \mathbb{Z}$$

associated to the rank map $K \rightarrow \mathbb{Z}$. The second desired statement follows from considering the fiber sequence

$$BSL \rightarrow BGL \rightarrow B_s \mathbb{G}_m$$

associated to the determinant map $BGL \rightarrow B_s \mathbb{G}_m$. $\square$

Theorem 2.6 (Theorem A). *For every prime p , there exists an $\mathbb{E}_3$ -motivic space map*

$$\Omega_{\mathbb{G}_m} BGL \rightarrow BGL_1(\mathrm{MGL})$$

whose associated $\mathbb{E}_3$ -MGL-motivic Thom spectrum is isomorphic to MF_p . Precomposition by $\Omega_{\mathbb{G}_m} BGL \rightarrow BGL_1(\mathrm{MGL})$ yields a $\mathbb{E}_3$ -motivic space map

$$\Omega_{\mathbb{G}_m} \mathrm{BSL} \rightarrow BGL_1(\mathrm{MGL})$$

whose associated $\mathbb{E}_3$ -MGL-motivic Thom spectrum is isomorphic to MZ .

Proof. Recall there exists an $\mathbb{E}_3$ -space map $U \rightarrow BGL_1(\mathrm{MU})$ whose $\mathbb{E}_3$ -MU-Thom spectrum is isomorphic to $\mathbb{F}_2$ and furthermore precomposing with $SU \hookrightarrow U$ yields an $\mathbb{E}_3$ -MU-space map $SU \rightarrow BGL_1(\mathrm{MU})$ whose $\mathbb{E}_3$ -MU-Thom spectrum is isomorphic to $\mathbb{Z}$ [Cra26, Theorem A]. These two statements are equivalent to the statements that there exists equivalences

$$\mathrm{MU} \otimes_{\mathrm{MU}[\Omega U]} \mathrm{MU} \rightarrow \mathbb{F}_p$$

and

$$\mathrm{MU} \otimes_{\mathrm{MU}[\Omega SU]} \mathrm{MU} \rightarrow \mathbb{Z}$$

of $\mathbb{E}_3$ -MU-algebras. Note that ΩU lifts to the motivic space K and $\Omega SU = BU$ lifts to the motivic space BGL . Both of these motivic spaces satisfy the hypothesis of Lemmas 2.3 and 2.4 therefore we get equivalences

$$\mathrm{MGL} \otimes_{\mathrm{MGL}[K]} \mathrm{MGL} \rightarrow \mathrm{MF}_p$$

and

$$\mathrm{MGL} \otimes_{\mathrm{MGL}[BGL]} \mathrm{MGL} \rightarrow \mathbb{Z}$$

of $\mathbb{E}_3$ -MGL-motivic Thom spectra. These statements are equivalent to the claims of existence of an $\mathbb{E}_3$ -motivic space map

$$B_s K \rightarrow BGL_1(\mathrm{MGL})$$

whose associated $\mathbb{E}_3$ -MGL-motivic Thom spectrum is isomorphic to MF_p and the existence of a $\mathbb{E}_3$ -motivic space map

$$B_s BGL \rightarrow BGL_1(\mathrm{MGL})$$

whose associated $\mathbb{E}_3$ -MGL-motivic Thom spectrum is isomorphic to MZ . The theorem follows by using Lemma 2.5 to rewrite $B_s K$ and $B_s BGL$. $\square$

Corollary 2.7 (Corollary 1.4). *Let a_i denote the generators of the Lazard ring $(\mathrm{MGL}_{(*,*)})_0$. Then there is an equivalence*

$$\mathrm{MGL} / (a_1, a_2, \dots) \simeq \mathrm{MZ}$$

of $\mathbb{E}_3$ -motivic spectra.

Proof. The map comes from the surjection $\mathrm{MGL}[BGL] \rightarrow \mathrm{MGL}$ that appeared from the equivalence

$$\mathrm{MGL} \otimes_{\mathrm{MGL}[BGL]} \mathrm{MGL} \rightarrow \mathbb{Z}$$

of $\mathbb{E}_3$ -MGL-motivic Thom spectra that appeared in the proof of Theorem . $\square$

3. NORMED MOTIVIC HOCHSCHILD HOMOLOGY

In this section we prove Theorem B. We begin by reviewing motivic Hochschild homology.

Definition 3.1 (Section 2.1 and Definition 4.1 of [BBDØ26]). *Let $X \in \mathrm{Spc}(X)$ and $A \rightarrow E$ a normed motivic spectrum map. Then $X \odot^A E$ is defined to be the tensor $X \otimes E$ in $\mathrm{NAlg}(\mathrm{Mod}_A(\mathrm{SH}(\mathbb{C}))) = \mathrm{NAlg}(\mathrm{SH}(\mathbb{C}))_{/A}$.*

Example 3.2. *Consider the normed motivic spectrum maps $\mathrm{MGL} \rightarrow \mathbb{Z}$ of [BH21, Example 16.34], $\mathbb{Z} \rightarrow \mathbb{F}_2$ of [BH21, Remark 14.6], and $\mathrm{MGL} \rightarrow \mathbb{F}_2$ formed by composition. We can thus consider $\mathbb{G}_m \otimes^{\mathrm{MGL}} \mathbb{F}_2$ and $\mathbb{G}_m \otimes^{\mathrm{MGL}} \mathbb{Z}$.*

To prove Theorem B, we will prove the existence of an $\mathbb{E}_2$ - MF_2 -algebra map $f_{\mathbb{F}_2}: \mathrm{MF}_2[\mathrm{BGL}] \rightarrow \mathbb{G}_m \odot^{\mathrm{MGL}} \mathrm{MF}_2$ and of an $\mathbb{E}_2$ - $\mathbb{Z}$ -algebra map. $f_{\mathbb{Z}}: \mathrm{MZ}[\mathrm{BSL}] \rightarrow \mathbb{G}_m \odot^{\mathrm{MGL}} \mathrm{MZ}$ before we prove that $f_{\mathbb{F}_2}$ and $f_{\mathbb{Z}}$ are equivalences. In [BBDØ26, Section 4.2], the authors construct a map from the free $\mathbb{E}_1$ - MF_2 -algebra $\mathrm{MF}_2[\mu_{2,1}]$ on a generator $\mu_{2,1}$ in bidegree $(2, 1)$ in the following way: to the class $\tau_0 \in \pi_{1,0}(\mathrm{MF}_2 \otimes \mathrm{MF}_2)$, they produce a class $\mu_{2,1} \in \pi_{2,1}(\mathbb{G}_m \odot \mathrm{MF}_2)$ via the map $\Sigma^{1,1} \mathrm{MF}_2 \otimes \mathrm{MF}_2 \rightarrow \mathbb{G}_m \odot \mathrm{MF}_2$. The class $\mu_{2,1} \in \pi_{2,1}(\mathbb{G}_m \odot \mathrm{MF}_2)$ determines an $\mathbb{E}_1$ - MF_2 -algebra map $\mathrm{MF}_2[\mu_{2,1}] \rightarrow \mathbb{G}_m \odot \mathrm{MF}_2$ by the universal property of $\mathrm{MF}_2[\mu_{2,1}]$.

We will form our $\mathbb{E}_2$ -algebra maps $f_{\mathbb{F}_2}$ and $f_{\mathbb{Z}}$ by a process that is significantly different from the way that the $\mathbb{E}_1$ - MF_2 -algebra map $\mathrm{MF}_2[\mu_{2,1}] \rightarrow \mathbb{G}_m \odot \mathrm{MF}_2$ above was formed. For context, we recall that [BBDØ26, Section 4] first form their map then determine upon Betti realization = inversion of τ that they recover Bockstein periodicity. To produce $f_{\mathbb{F}_2}$ and $f_{\mathbb{Z}}$, we will “do things the other way around”. We will start with desired underlying topological equivalences $\mathrm{THH}(\mathbb{F}_2/\mathrm{MU}) \simeq \mathbb{F}_2[B\mathrm{U}]$ of $\mathbb{E}_2$ - $\mathbb{F}_p$ -algebras and $\mathrm{THH}(\mathbb{Z}/\mathrm{MU}) \simeq \mathbb{Z}[BS\mathrm{U}]$ of $\mathbb{E}_2$ - $\mathbb{Z}$ -algebras and deduce that there exist unique choices of $f_{\mathbb{F}_2}$ and $f_{\mathbb{Z}}$ that motivically enhance these topological equivalences. The uniqueness of such choices will come from cellularity and Tateness forcing fully faithfulness of Betti realization. The precise content of this idea is stated in Proposition 3.13.

Before we state Proposition 3.13, we make our setting modulo τ .

Notation 3.3. *We define $\overline{\mathrm{MF}_2} := \mathrm{MF}_2/\tau$ and $\overline{\mathrm{MZ}_2^\wedge} := \mathrm{MZ}/\tau$.*

Theorem 3.4 (Theorem 2.9 of [BBDØ26], forthcoming work of Bachmann-Hopkins). *The motivic spectra $\overline{\mathrm{MF}_2}$ and $\overline{\mathrm{MZ}_2^\wedge}$ admit normed refinements.*

The equivalence in Theorem A of $\mathbb{F}_p$ being Thom spectrum of the $\mathbb{E}_3$ -map $U \rightarrow \mathrm{BGL}_1(\mathrm{MU})$ and Proposition ?? give equivalences

Lemma 3.5. *There exist equivalence*

$$(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathrm{MF}_2) \otimes_{\mathrm{MGL}} \overline{\mathrm{MF}_2} \simeq \mathbb{G}_m \odot^{\overline{\mathrm{MF}_2}} \overline{\mathrm{MF}_2} \otimes_{\mathrm{MGL}} \mathrm{MF}_2$$

of normed motivic spectra. Furthermore there exists an equivalence

$$(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathrm{MF}_2) \otimes_{\mathrm{MF}_2} \overline{\mathrm{MF}_2} \simeq \mathbb{G}_m \odot^{\overline{\mathrm{MF}_2}} \overline{\mathrm{MF}_2} \otimes_{\mathrm{MGL}} \mathrm{MF}_2 \otimes_{\overline{\mathrm{MF}_2} \otimes_{\mathrm{MGL}} \mathrm{MF}_2} \overline{\mathrm{MF}_2}$$

of normed motivic spectra.

Proof. This follows from standard properties of $\mathbb{G}_m \odot -$ and of normed spectra. $\square$

To work modulo τ , we tensor with $\overline{\mathrm{MF}_2}$ so that we seek to produce an $\mathbb{E}_2$ - $\overline{\mathrm{MF}_2}$ -algebra map

$$\overline{\mathrm{MF}_2}[\mathrm{BGL}] \rightarrow \mathbb{G}_m \odot^{\overline{\mathrm{MF}_2}} \overline{\mathrm{MF}_2} \otimes_{\mathrm{MF}_2} \overline{\mathrm{MF}_2}$$

Recollection 3.6. We recall several facts from [BBDØ26]. The objects of $\mathrm{NAlg}(\mathrm{SH}(\mathbb{C}))_{\overline{\mathrm{MF}_2}}$ have Voevodsky/topological operations Q_i and higher divided power operations δ_i . We use the indexing conventions established in [BBDØ26, Section 3].

There is a sequence of forgetful functors

$$\mathrm{NAlg}(\mathrm{SH}(\mathbb{C})) \rightarrow \mathcal{NA} \rightarrow \mathcal{DA} \rightarrow \mathrm{Vect}_{\mathbb{F}_2}^{\mathbb{Z} \times \mathbb{Z}}$$

studied in [BBDØ26, Section 4] with the forgetful functor $\mathrm{NAlg}(\mathrm{SH}(\mathbb{C})) \rightarrow \mathcal{NA}$ is given by forgetting the operations Q_i for $i > 2$, the forgetful functor $\mathcal{NA} \rightarrow \mathcal{DA}$ is given by forgetting the operations Q_2 and Q_1 , and the forgetful functor $\mathcal{DA} \rightarrow \mathrm{Vect}_{\mathbb{F}_2}^{\mathbb{Z} \times \mathbb{Z}}$ is given by forgetting all operations thus is given by taking underlying bigraded $\mathbb{F}_2$ -vector space.

Lemma 3.7. We describe the MF_2 -module $\overline{\mathrm{MF}_2} \otimes_{\mathrm{MGL}} \mathrm{MF}_2$ as an object of $\mathcal{NA}$. We have

$$\pi_{*,*}^{\mathcal{NA}}(\overline{\mathrm{MF}_2} \otimes_{\mathrm{MGL}} \mathrm{MF}_2) \cong \Lambda_{\mathbb{F}_2}(\sigma a_0, \sigma a_1, \dots)$$

with $\deg(\sigma a_i) = (2i+1, 1)$ for all $i \geq 0$. Furthermore we have $Q_1 = 0$ and that $Q_2(e_i) = e_{2i+1}$ for all $i \geq 0$.

Proof. By the Hopkins-Morel-Hoyois theorem, $\mathbb{F}_2$ is a regular quotient of MGL , thus the $\pi_{*,*}^{\mathcal{NA}}$ calculation can be seen by resolving $\mathbb{F}_2$ as a MGL -module by an appropriate Koszul resolution. We have $Q_1 = 0$ for degree reasons so it remains to justify the Q_2 calculation.

To see the Q_2 calculation, we first recall from [Koc71] that on Betti realization we have Q_2 on $\mathbb{F}_2[BU]$ is given by $Q_1(e_i) = e_{2i+1} \bmod$ decomposables. By naturality of power operations we conclude the same for the MGL -dual steenrod algebra, and once again by naturality of power operations, we conclude the same modulo τ . Finally we observe there cannot be decomposable for bidegree reasons. $\square$

Recollection 3.8. We recall the following from [BBDØ26].

- There exists a derivation d of bidegree $(1, 1)$ coming from $\mathbb{G}_m$ -suspension.
- There is functor $G : \mathcal{NA} \rightarrow \mathcal{DA}$ such that

$$G(F(W)) \simeq UF(W) \otimes F_{\mathcal{DA}}(dW)$$

- There if a functor $DG' : D(\mathcal{NA}) \rightarrow D(\mathcal{DA})$ given by first taking simplicial free $\mathcal{NA}$ -resolution then applying B .

Lemma 3.9. There exists a strongly convergent spectral sequence

$$\pi_s \left(\mathbb{F}_2 \otimes_{U\Lambda_{\mathbb{F}_2}}(\sigma a_0, \sigma a_1, \dots) DG'(\Lambda_{\mathbb{F}_2}(\sigma a_0, \sigma a_1, \dots)) \right)_{t,w} \Rightarrow \pi_{s+t,w}((\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{F}_2)\tau)$$

There is a similar statement for $\mathbb{Z} \bmod (2, \tau)$.

Proof. This follows from [BBDØ26, Section 4.5] $\square$

Note that there is a decomposition

$$\Lambda_{\mathbb{F}_2}(\sigma a_0, \sigma a_1) = \bigotimes_n \mathrm{Free}_{Q_2}(\sigma a_{4n})$$

for MF_2 . For MZ , there is one different in that the $\mathrm{Free}_{Q_2}(\sigma a_{2n})$ starts with σa_1 instead of σa_0 , so we get the decomposition

$$\Lambda_{\mathbb{Z}}(\sigma a_1) = \mathrm{Free}_{Q_2}(\sigma a_2) \otimes \bigotimes_{n>0} u\mathrm{Free}_{Q_2}(\sigma a_{4n})$$

We can further describe $\text{Free}_{Q_2}(\sigma a_{2n})$ following [BBDØ26, Remark 4.8]. The free object $F(\mu_{m,n})$ consists of all the $Q_2^n \mu_{m,n}$ as well as all the $Q_1 Q_2^n(\mu_{m,n})$. This suggests $\text{Free}_{Q_2}(\sigma a_{2n})$ should be thought of as $F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))$ which guides us in making our derived models.

Lemma 3.10. *There exists a free $\mathcal{NA}$ -resolution of $\text{Free}_{Q_2}(\sigma a_{2n})$ given by the following complex: let there be exactly one module $\mathbb{F}_2\{x_m\}$ in each degree $(2^{m+1}n - 1 - m, 2^m n - 1)$, which one should think of as $(Q_2)^m \sigma a_{2n}$ and define the differential on x_m as $Q_1 x_{m-1}$.*

Proof. By exactness, this complex preserves all $(Q_2)^m \sigma a_{2n}$ while killing $Q_1(\sigma a_{2n})$ thus producing a resolution of $F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))$. $\square$

Now that we have made a free $\mathcal{NA}$ -resolution of $F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))$, an object of $D(\mathcal{NA})$, we have made some progress towards understanding the object

$$\mathbb{F}_2 \otimes_{U_{\Lambda_{\mathbb{F}_2}}(\sigma a_0, \sigma a_1, \dots)} (DG'(F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))))$$

in $D(\mathcal{DA})$. Recall that DG' is defined by applying G to the free $\mathcal{NA}$ -resolution we calculated above. We begin with calculating the homology associated to it as this calculation is determined by the operations.

Lemma 3.11. *The trigraded homotopy of*

$$\mathbb{F}_2 \otimes_{U_{\Lambda_{\mathbb{F}_2}}(\sigma a_0, \sigma a_1, \dots)} (DG'(F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))))$$

is

$$\mathbb{F}_2[dx_0] \otimes \otimes_{k \geq 0} \Gamma_{\mathbb{F}_2} y_{n,k}$$

with $|dx_0| = (0, 2n, n)$ and $|y_{n,k}| = (1, 2^{k+2}n - 1, 2^{k+1}n)$.

Proof. Firstly we explain the effects of G : the x_k 's become replaced with dx_k where $|dx_k| = (2^{k+1}n - k, 2^k n)$ because d is of bidegree $(-1, -1)$. Furthermore the differentials become $\delta(dx_k) = Q_{-1}x_{k-1}$ $dQ_1 = Q_{-1}d$. Note that $\delta_0 = 0$ so H_0 is just made up of the Chow degree 0 elements as desired. To see the divided power algebra parts, one argues in a similar way to the proof of [BBDØ26, Theorem 3.19], $\square$

Now that we calculated the homology of

$$\text{MF}_2 \otimes_{U(F(\sigma a_{2n})/(Q_1(\sigma a_{2n})))} (DG')F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))$$

we are ready to realize it in Lemma 3.12.

Lemma 3.12. *The object*

$$\text{MF}_2 \otimes_{U(F(\sigma a_{2n})/(Q_1(\sigma a_{2n})))} (DG')F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))$$

of $D(\mathcal{DA})$ is equivalent to

$$\text{MF}_2[dx_0] \otimes \text{Bar}_*(\text{MF}_2, \Lambda_{\mathbb{F}_2}(y_0, y_1, \dots), \text{MF}_2)$$

with $\deg(dx_0) = (2n, n)$ and $\deg(y_k) = (2^{k+2}n - 1, 2^{k+1}n)$ for all $k \geq 0$. Note that we suppressed the third grading coming from Bar .

Proposition 3.13. *The $\overline{\text{MF}}_2$ -modules $(\mathbb{G}_m \odot^{\text{MGL}} \mathbb{F}_2) / \tau$ and $(\mathbb{G}_m \odot^{\text{MGL}} \mathbb{MZ}) / (2, \tau)$ are Tate.*

Proof. Consider the spectral sequence of Lemma 3.9. By evenness and tridegree reasons, the spectral sequence collapses on the $\mathbb{E}_2$ -page with no extension problems, therefore it suffices to show Tateness and cellularity

$$\mathbb{F}_2 \otimes_{U_{\Lambda_{\mathbb{F}_2}}(\sigma a_0, \sigma a_1, \dots)} (DG'(\Lambda_{\mathbb{F}_2}(\sigma a_0, \sigma a_1, \dots))).$$

Note that this split up, via the $\Lambda_{\mathbb{F}_2}(\sigma a_0, \sigma a_1, \dots)$ as a tensor product of the $F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))$'s. Tensor product preserves Tateness and cellularity, and so do the processes of DG' therefore it suffices to show Tateness and cellularity for the individual

$$\mathrm{MF}_2 \otimes_{U(F(\sigma a_{2n})/(Q_1(\sigma a_{2n})))} (DG')F(\sigma a_{2n})/(Q_1(\sigma a_{2n}))'s.$$

We showed this in Lemma 3.12. The $\mathbb{Z}$ case follows from the minor variation previously mentioned. $\square$

Now that we've completed the hard work of showing the Tateness and cellularity of the $\overline{\mathrm{MF}}_2$ -modules $(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{F}_2)\tau$ and $(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{Z})/(2, \tau)$, we move on to showing that we have Tateness and cellularity before modding out τ .

Lemma 3.14. *There are $\mathbb{E}_2$ -MU-algebra equivalence*

$$\mathrm{Re}(\mathbb{G}_m \odot^{\mathrm{MGL}_p^\wedge} \mathrm{MF}_2[\tau^{-1}]) \simeq \mathrm{THH}(\mathbb{F}_2/\mathrm{MU}) \rightarrow \mathbb{F}_2[\mathrm{BU}] \simeq \mathrm{Re}(\mathrm{MF}_2[BGL])$$

Lemma 3.15. *The Tate $\overline{\mathrm{MF}}_2$ -modules $\overline{\mathrm{MF}}_2$ -module $\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{F}_2/\tau$ and $(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{Z})/(2, \tau)$ are pure Tate. In particular*

$$\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{F}_2 \simeq \bigoplus_{n \geq 0} (\Sigma^{2n, n} \mathrm{MF}_2)^{\oplus p(n)}$$

and

$$(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{Z})/2 \simeq \bigoplus_{n \geq 0} (\Sigma^{2n, n} \mathrm{MF}_2)^{\oplus q(n)}$$

each as MF_2 -modules where $p(n)$ denotes the partition function and $q(n)$ denotes the reduced partition function $p(n) - p(n-1)$.

Proof. We've already established Tateness so it remains to justify the pureness and the ranks $p(n)$ and $q(n)$. This follows from a counting argument given Lemma 3.11. $\square$

Lemma 3.16. *The MF_2 -modules $(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{F}_2)$ and $(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathrm{MZ})/2$ are pure Tate.*

Proof. We first argue that $\pi_{s,t} = 0$ for these two MF_2 -modules whenever $s < t$. This follows from MGL , MZ , MF_2 being very effective ([?, Section 5], [Hoy15, Section 7]) and that being very effective is preserved by the various operations we apply. To deduce purity and the MF_2 -module structures, this from looking at the cofiber sequences associated to τ and using the vanishing above argued from very effectiveness. $\square$

Corollary 3.17. *The motivic spectra $\mathrm{MGL}[BGL]$ and $\mathrm{MGL}[BSL]$ are pure Tate of finite type and have the same MF_2 -module decomposition as $\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{F}_2$ and $(\mathbb{G}_m \odot^{\mathrm{MGL}} \mathbb{Z})$, respectively.*

Lemma 3.18. *Betti realization is fully faithful on nonnegative, finite-type pure Tate MF_2 -modules, and a similar statement is true for MZ -modules*

Proof. To show the claim MF_2 -modules, by our assumptions it suffices to show this claim for the MF_2 -modules $\sigma^{n, 2n} \mathrm{MF}_2$. This case was proven in [BBDØ26, middle of page 6]. The argument for MZ -modules is essentially the same starting with the fact that $\pi_{*,*}(\mathrm{MZ}) \cong \mathbb{Z}[\tau]$. $\square$

Theorem 3.19 (Theorem B). *There exists an equivalence*

$$\mathbb{G}_m \odot^{\mathrm{MGL}} \mathrm{MF}_2 \simeq \mathrm{MF}_2[BGL]$$

of $\mathbb{E}_2$ - MF_2 -algebras. Furthermore there exists an equivalence

$$\mathbb{G}_m \odot^{\mathrm{MGL}} \mathrm{MZ} \simeq \mathrm{MZ}[BSL]$$

of $\mathbb{E}_2$ - MZ -algebras.

Proof. We use Lemma 3.18 to motivially lift the equivalences $\mathrm{THH}(\mathbb{F}_2/\mathrm{MU}) \cong \mathbb{F}_2[\mathrm{BU}]$ and $\mathrm{THH}(\mathbb{Z}/\mathrm{MU}) \cong \mathbb{Z}[\mathrm{BSU}]$ of $\mathbb{E}_2$ -rings from topology by Lemma 3.18. $\square$

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