SYNTOMIC COHOMOLOGY OF $\mathbb{Z}$ AND ℓ

PRESTON CRANFORD

Abstract. These are notes for Talk 4.4 of Talbot 2024 that I delivered. We calculate the mod (p, v_1) syntomic cohomology of $\mathbb{Z}_{(p)}$ and the mod (p, v_1, v_2) syntomic cohomology of $\ell_p^\wedge$. Then we discuss features of these calculations including Tate duality, collapsing Bockstein spectral sequence, and Lichtenbaum–Quillen conjecture.

In this talk we calculate the (p, v_1, v_2) syntomic cohomology of $\ell_p^\wedge$. Then we discuss three features of the calculation: Tate duality, collapsing of v_2-Bockstein spectral sequence, and satisfaction of the Lichtenbaum–Quillen conjecture.

Fix a prime p. Let ℓ denote the Adams summand $\ell_p^\wedge$, the ring spectrum with its usual $\mathbb{E}_\infty$-structure.

Before we calculate the (p, v_1, v_2) syntomic cohomology of $\ell_p^\wedge$, we start by defining syntomic cohomology of suitable $\mathbb{E}_\infty$-rings, chromatically p-quasisyntomic $\mathbb{E}_\infty$-rings. For the statement of the definition we recall the following definition.

Definition 1. [HRW, Section 1] *For R a chromatically p-quasisyntomic $\mathbb{E}_\infty$-ring, the syntomic cohomology of R is the associated graded of* $\mathrm{fil}_{\mathrm{mot}}\,\mathrm{TC}(R)_p^\wedge = \mathrm{fil}_{ev,p,TC}\,\mathrm{THH}(R)_p^\wedge$.

Now that we have defined syntomic cohomology of chromatically p-quasisyntomic $\mathbb{E}_\infty$-rings, we define mod $(p, v_1, \ldots, v_n)$ syntomic cohomology to be the mod $(p, v_1, \ldots, v_n)$ reduction of syntomic cohomology.

Recalling from a previous talk that $\ell_p^\wedge$ is chromatically p-quasisyntomic, we now begin calculating its mod (p, v_1, v_2) syntomic cohomology. Since we've already calculated what the canonical map

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p, v_1, v_2) \xrightarrow{\text{"}can\text{"}} \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p, v_1, v_2)$$

looks like on the level of homotopy groups in Talk 4.3, it only remains to understand what the Frobenius map is on the level of homotopy groups. The main idea of the proof is to show that the mod (p, v_1, v_2) Frobenius

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p, v_1, v_2) \xrightarrow{\varphi} \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p, v_1, v_2)$$

is given by the mod (p, v_1) Frobenius

$$\left(\mathrm{THH}(\ell)\right)/(p, v_1) \xrightarrow{\varphi} \left(\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1)$$

as we recall now.

Lemma 2 (Talk 4.1)**.** *The mod $(p, v_1, \ldots, v_n)$ Frobenius map $\pi_*\left(\mathrm{THH}(BP\langle n\rangle)\otimes_{BP\langle n\rangle}\mathbb{F}_p\right) \xrightarrow{\varphi} \pi_*\left(\mathrm{THH}(BP\langle n\rangle)^{\mathrm{tC}_p}\otimes_{BP\langle n\rangle}\mathbb{F}_p\right)$ is identified with the ring map*

$$\Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n)\otimes\mathbb{F}_p[\mu] \to \Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n)\otimes\mathbb{F}_p[\mu^{\pm 1}]$$

that inverts the class μ.

Now that we've recalled Lemma 2 just for the case $n = 1$, so that $BP\langle n\rangle = \ell$, we are ready to compute the mod (p, v_1, v_2) Frobenius in Lemma 3.

Lemma 3. [HRW, Corollary 6.6.1] *In terms of the isomorphisms*

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p, v_1, v_2) \cong \mathbb{F}_p[t^{p^2}, \mu]/(t^{p^2}\mu) \otimes \Lambda(\lambda_1, \lambda_2) \oplus \mathbb{F}_p\{t^d\lambda_1, t^{pd}\lambda_2, t^d\lambda_1\lambda_2, t^{pd}\lambda_1\lambda_2 \mid 0 < d < p\},$$

$$\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p, v_1, v_2) \cong \mathbb{F}_p[t^{\pm p^2}] \otimes \Lambda(\lambda_1, \lambda_2)$$

calculated in a previous talk, the Frobenius is trivial on classes not of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ where $k \geq 0$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and sends each class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ to an $\mathbb{F}_p^\times$ multiple of the class named $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{-p^2k}$.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{TC}^-(\ell) \arrow[r, "\varphi"] \arrow[d] & \mathrm{TP}(\ell) \arrow[d] \\
\mathrm{THH}(\ell) \arrow[r, "\varphi"] & \mathrm{THH}(\ell)^{\mathrm{tC}_p}
\end{tikzcd}$$

with the left vertical arrow given by the unit map. The diagram commutes by the Tate orbit lemma. Modding out by (p, v_1) gives the commuting diagram

$$\begin{tikzcd}
\left(\mathrm{TC}^-(\ell)\right)/(p, v_1) \arrow[r, "\varphi"] \arrow[d] & \left(\mathrm{TP}(\ell)\right)/(p, v_1) \arrow[d] \\
\left(\mathrm{THH}(\ell)\right)/(p, v_1) \arrow[r, "\varphi"] & \left(\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1).
\end{tikzcd}$$

Since we understand the Frobenius bottom horizontal map, we have an opportunity to understand the Frobenius top horizontal map. Using that the Frobenius preserves the motivic filtration as stated in the following black-boxed lemma:

Lemma 4. [HRW, Theorem 1.3.6] *Let R be a chromatically quasisyntomic $\mathbb{E}_\infty$-ring. Then, for each prime number p, the Nikolaus–Scholze Frobenius*

$$\varphi : \mathrm{TC}^-(R)^\wedge_p \to \mathrm{TP}(R)^\wedge_p$$

refines to a natural map

$$\varphi : \mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(R)^\wedge_p \to \mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TP}(R)^\wedge_p.$$

The same is true of the canonical map between the same objects, and $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$ can be computed as the equalizer of the filtered Frobenius and canonical maps.

and taking appropriate reductions, we would get the following commutative diagram

$$\begin{tikzcd}
\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell)\right)/(p, v_1) \arrow[r, "\varphi"] \arrow[d] & \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell)\right)/(p, v_1) \arrow[d] \\
\left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell)\right)/(p, v_1) \arrow[r, "\varphi"] & \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1).
\end{tikzcd}$$

We argue that the above diagram factors through one of the form

$$\begin{tikzcd}
\left(\mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{TC}^{-}(\ell)\right)/(p, v_1, v_2) \arrow[r, "\varphi"] \arrow[d, "f"] & (\mathrm{gr}^{*}_{\mathrm{mot}} \mathrm{TP}(\ell))/(p, v_1, v_2) \arrow[d, "g"] \\
(\mathrm{gr}^{*}_{\mathrm{mot}} \mathrm{THH}(\ell))/(p, v_1) \arrow[r, "\varphi"] & \left(\mathrm{gr}^{*}_{\mathrm{mot}} \mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1).
\end{tikzcd}$$

Note that $v_2 = 0$ in $(\mathrm{gr}^{\star}_{\mathrm{ev}}\, \ell)/(p, v_1)$, so the sequence of algebra maps

$$\mathrm{gr}^{\star}_{\mathrm{ev}}\, \ell \to \mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{THH}(\ell) \xrightarrow{\varphi} \mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{THH}(\ell)^{\mathrm{tC}_p}$$

imply that $v_2 = 0$ in $\left(\mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1)$. Thus, the natural map

$$\mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{TP}(\ell)/(p, v_1) \to \mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{THH}(\ell)^{\mathrm{tC}_p}/(p, v_1)$$

factors over a map

$$g : \mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{TP}(\ell)/(p, v_1, v_2) \to \mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{THH}(\ell)^{\mathrm{tC}_p}/(p, v_1).$$

Now that we have the Frobenius map we wish to understand in a commuting diagram with a Frobenius map we understand, we now start studying it. We leave the proof of g being an isomorphism to the next Lemma 5. The map f is trivial on every class not of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$. Lemma 2 and g is an isomorphism (Lemma 5) imply that each class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ has non-trivial Frobenius image. The only non-trivial classes in the codomain in the same degree as $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$, are $\mathbb{F}_p^{\times}$ multiples of the class named $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{-p^2k}$. □

The above proof is complete except for the justification of the map g being an isomorphism. We justify this now.

Lemma 5. [HRW, Theorem 6.4.1] *The map g above is an isomorphism*

$$(\mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{TP}(\ell))/(p, v_1, v_2) \xrightarrow{\cong} \left(\mathrm{gr}^{\star}_{\mathrm{mot}}\, \mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1)$$

Proof. We can compute the motivic associated graded for $\mathrm{TP}(\ell)$ via the cobar complex associated to the descent

$$TP(\ell) \to \mathrm{TP}(\ell/\,\mathrm{MU})$$

with sth term given by $\pi_*\mathrm{TP}(\ell/MU^{s+1})$. The domain of g is calculated by the complex obtained from modding out each term $\pi_*\mathrm{TP}(\ell/\mathrm{MU}^{\otimes s+1})$ by p, v_1 and v_2. The codomain is obtained from the complex $\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes \bullet+1})^{\mathrm{tC}_p}$ by levelwise killing p and v_1. We first note that, for each value of $s \geq 0$, $v_2 = 0$ in $\left(\pi_*\mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p, v_1)$. This can be seen from the existence of the relative cyclotomic Frobenius map

$$\pi_*\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})/(p, v_1) \to \pi_*\mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}/(p, v_1),$$

because $v_2 \equiv 0$ modulo (p, v_1) in any $\pi_*\ell$-algebra such as $\pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})$. It follows that g extends to a map of cobar complexes which levelwise is of the form

$$g_s : \left(\pi_*\mathrm{TP}(\ell/\,\mathrm{MU}^{\otimes s+1})\right)/(p, v_1, v_2) \to \left(\pi_*\mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p, v_1).$$

To prove that g is an isomorphism, we will prove the stronger claim that g_s is an isomorphism for each $s \geq 0$. To see that each g_s is an isomorphism we first use Lemma 6 below to see that the group $\left(\pi_*\mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p, v_1, v_2)$ can be computed from $\left(\pi_*\mathrm{TP}(\ell/\,\mathrm{MU}^{\otimes s+1})\right)/(p, v_1)$ by killing $[p](t)$ for any complex orientation t.

Lemma 6. [HRW, Lemma 6.4.2] *Let $M \in \mathrm{Mod}_{\mathrm{MU}}^{\mathrm{BS}^1}$ be an* S^1*-equivariant* MU*-module. Then the map*

$$M^{\mathrm{tS}^1}/[p](t) = M^{\mathrm{tS}^1} \otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p} \to M^{\mathrm{tC}_p}$$

is an equivalence. In particular for $\left(\pi_* \mathrm{THH}(\ell/ MU^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p, v_1) \in \mathrm{Mod}_{\mathrm{MU}}^{\mathrm{BS}^1}$, we have an equivalence

$$\left(\pi_* \mathrm{THH}(\ell/ MU^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p, v_1) \cong \left(\left(\pi_* \mathrm{THH}(\ell/ MU^{\otimes s+1})^{\mathrm{tS}^1}\right)/(p, v_1)\right)/([p](t))$$

Proof. We argue exactly as in [NS, IV.4.12]. Since $\mathrm{MU}^{\mathrm{tC}_p} = \mathrm{MU}^{\mathrm{tS}^1}/[p](t)$ is a perfect $\mathrm{MU}^{\mathrm{tS}^1}$-module as seen in the following sequence

$$\mathrm{MU}^{\mathrm{tC}_p} \to \mathrm{MU}^{\mathrm{tS}^1} \xrightarrow{[p](t)} \mathrm{MU}^{\mathrm{tS}^1},$$

the functor $(-) \otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p}$ commutes with all limits and colimits. Using the equivalences $M^{tG} = \mathrm{colim}(\tau_{\geq n} M)^{tG}$ and $M^{tG} = \lim(\tau_{\leq n} M)^{tG}$ for $G = C_p$ it is sufficient to assume that M is bounded. By filtering M by its Postnikov filtration it is sufficient to assume that M is discrete. For M is discrete then its S^1-action is trivial because

$$Fun(BS^1, Aut(M)) = Fun(BS^1, Aut(\pi_0(M))$$

and BS^1 is connected. Now that we've reduced to the discrete S^1-action case, we show for M with a trivial S^1-action that

$$M^{\mathrm{tS}^1}/[p](t) = M^{\mathrm{tS}^1} \otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p} \to M^{\mathrm{tC}_p}$$

as desired. Since M has a trivial S^1-action we have that $A^{hS^1} = A^{BS^1}$. Since M has a trivial S^1-action, M has a trivial C_p-action, so likewise $A^{hC_p} = A^{BC_p}$. Consider the fiber sequence

$$S^1 \to BC_p \to BS^1 \xrightarrow{d \to d^{\otimes p}} BS^1.$$

of spaces. Dualizing and lifting to spectra gives a cofiber (=fiber) sequence

$$\Sigma^\infty BC_p \to \Sigma^\infty BS^1 \to (BS^1)^\wedge.$$

Homming from M then determines a fiber (=cofiber) sequence

$$M^{BC_p} \leftarrow M^{BS^1} \leftarrow M^{(BS^1)^\wedge}.$$

Because M is a MU-module, there is a Thom isomorphism $M^{(BS^1)^\wedge} \cong \Sigma^{-2} M^{BS^1}$. Note that

$$(BS^1)^\wedge = \mathrm{cofib}(S(V) \to D(V))$$

and $D(V) \cong \mathrm{CP}^\infty$. Finally, the map

$$\Sigma^{-2} M^{BS^1} \to M^{BC_p}$$

can be identified with the map $M[[t]] \to M[[t]]$ sending 1 to the Euler characteristic $p[t]$. This proves

$$M^{h\mathrm{S}^1}/[p](t) = M^{h\mathrm{S}^1} \otimes_{\mathrm{MU}^{h\mathrm{S}^1}} \mathrm{MU}^{h\mathrm{C}_p} \to M^{h\mathrm{C}_p}$$

is an equivalence, and the analogous statement with $\mathrm{t}S^1$ and $\mathrm{t}C_p$ in place of $h\mathrm{S}^1$ and $\mathrm{t}C_p$, respectively, follows from inverting t.

$\square$

We finish by using the black-boxed Lemma 7

Lemma 7. [HRW, Section 6] *We have that v_2 is a unit multiple of $[p](t)$.*

Finally knowing that $v_2 = cp[t](t)$ for a unit c allows us to conclude that

$$\begin{aligned}\left(\left(\pi_* \mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1)\right)/([p](t)) &= \left(\left(\pi_* \mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1)\right)/(cv_2)\\ &= \left(\left(\pi_* \mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1)\right)/(v_2)\\ &= \left(\left(\pi_* \mathrm{THH}(\ell/\,\mathrm{MU}^{\otimes s+1})^{\mathrm{t}S^1}\right)/(p,v_1,v_2)\right)\end{aligned}$$

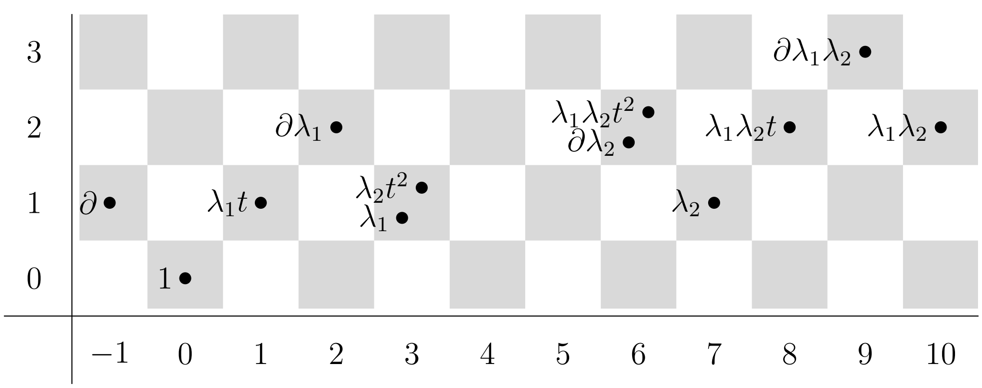
as desired. $\square$

Now that we know the Frobenius, we are ready for the syntomic cohomology calculation.

Theorem 8. [HRW, Theorem 6.0.4] *The mod (p, v_1, v_2) syntomic cohomology of ℓ is a finite $\mathbb{F}_p$-vector space. As a vector space, it is isomorphic to*

(1) $\mathbb{F}_p\{1\}$, *in Adams weight* 0 *and degree* 0.
(2) $\mathbb{F}_p\{\partial, t^d\lambda_1, t^{dp}\lambda_2 \mid 0 \le d < p\}$, *in Adams weight* 1. *Here,* $|\partial| = -1$, $|t^d\lambda_1| = 2p-2d-1$, *and* $|t^{dp}\lambda_2| = 2p^2-2dp-1$.
(3) $\mathbb{F}_p\{t^d\lambda_1\lambda_2, t^{dp}\lambda_1\lambda_2, \partial\lambda_1, \partial\lambda_2 \mid 0 \le d < p\}$, *in Adams weight* 2. *Here,* $|t^d\lambda_1\lambda_2| = 2p^2-2p-2d-2$, $|t^{dp}\lambda_1\lambda_2| = 2p^2-2p-2dp-2$, $|\partial\lambda_1| = 2p-2$, *and* $|\partial\lambda_2| = 2p^2-2$.
(4) $\mathbb{F}_p\{\partial\lambda_1\lambda_2\}$, *in Adams weight* 3 *and degree* $2p^2+2p-3$.

The mod $(2, v_1, v_2)$-syntomic cohomology of $\ell = \mathrm{ku}$



Proof. By definition of TC, we have the fiber sequence

$$\mathrm{TC}(\ell) \to \mathrm{TC}^-(\ell) \xrightarrow{\varphi - \mathrm{can}} \mathrm{TP}(\ell),$$

and since the motivic filtration preserving the fiber sequence, we have a fiber sequence

$$\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell) \to \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell) \xrightarrow{\varphi - \mathrm{can}} \mathrm{gr}_{\mathrm{mot}} * \mathrm{TP}(\ell).$$

By taking appropriate reductions, we have the induced fiber sequence

$$\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell)/(p,v_1,v_2) \to \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p,v_1,v_2) \xrightarrow{\varphi-\mathrm{can}} \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2).$$

which induces a long exact sequence

$$\begin{aligned}\cdots \to \mathrm{gr}^{*-1}_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2) &\to \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\ell)/(p,v_1,v_2)\\ \to \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p,v_1,v_2) &\xrightarrow{\varphi-\mathrm{can}} \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2) \to \cdots\end{aligned}$$

of groups which we will use to deduce the result. The map

$$\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p,v_1,v_2) \xrightarrow{\varphi-\mathrm{can}} \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2)$$

is given by the rule that classes of the form $\lambda_1^{e_1}\lambda_2^{e_2}t^{kp^2} \in \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p,v_1,v_2)$ go to $-\lambda_1^{e_1}\lambda_2^{e_2}t^{kp^2} \in \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2)$, classes of the form $\lambda_1^{e_1}\lambda_2^{e_2}t\mu^k \in \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p,v_1,v_2)$ go to a $\mathbb{F}_p^\times$-multiple of $\lambda_1^{e_1}\lambda_2^{e_2}t^{-kp^2} \in \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2)$, and all other classes go to zero. The kernel of this map gives the terms in the syntomic cohomology calculation that don't have λ_i's in them. The cokernel is given by

$$\frac{\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2)}{\mathrm{im}(\varphi-\mathrm{can})} \cong \frac{\mathbb{F}_p[t^{p^2}]\otimes\Lambda(\lambda_1,\lambda_2)}{F_p[t^{p^2}]} \cong \Lambda(\lambda_1,\lambda_2).$$

This contributes the basis elements with a ∂ coming from the degree shift. □

Now we discuss three consequences of the syntomic cohomology calculation. The first is the following.

Proposition 9. [HRW, Theorem 6.0.4] *The v_2-Bockstein spectral sequence (converging to the mod (p,v_1) syntomic cohomology of ℓ as an $\mathbb{F}_p[v_2]$-module) collapses with no differentials. As a consequence, we've calculated $\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p,v_1)$ as an $\mathbb{F}_p[v_2]$-vector space.*

Proof. Recall that the v_2-Bockstein spectral sequence (converging to the mod (p,v_1) syntomic cohomology of ℓ as an $\mathbb{F}_p[v_2]$-module) has signature

$$\pi_{*,*}\left(\left(gr^*_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p,v_1)\right)/(v_2)\right)[v_2] \to \left(gr^*_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p,v_1)\right)$$

which can be rewritten as

$$\pi_{*,*}\left(gr^*_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p,v_1,v_2)\right)[v_2] \to \left(gr^*_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p,v_1)\right)$$

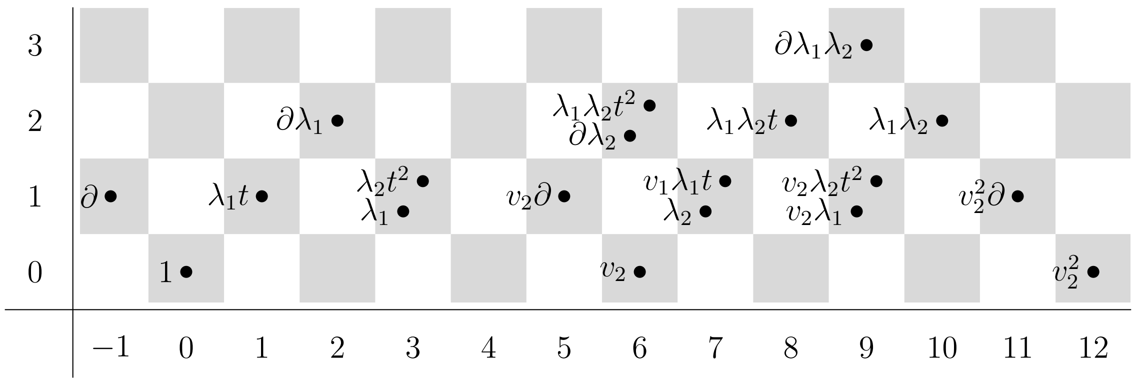
due to regularity. Since we know $gr^*_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p,v_1,v_2)$ and that $|v_2| = (2p^2-2,0)$ we have the following description of the E_1 page of the spectral sequence.

We have $\pi_{*,*}\left(gr^*_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p,v_1,v_2)\right)[v_2]$ as an $\mathbb{F}_p$ vector space isomorphic to

(1) $\mathbb{F}_p\{v_2^e \mid 0 \le d < p, e \ge 0\}$, in Adams weight 0 and degree $(2p^2-p)e$.
(2) $\mathbb{F}_p\{v_2^e\partial, v_2^e t^d\lambda_1, v_2^e t^{dp}\lambda_2 \mid 0 \le d < p, e \ge 0\}$, in Adams weight 1. Here, $|\partial| = -1$, $|t^d\lambda_1| = 2p-2d-1$, and $|t^{dp}\lambda_2| = 2p^2-2dp-1$.
(3) $\mathbb{F}_p\{v_2^e t^d\lambda_1\lambda_2, v_2^e t^{dp}\lambda_1\lambda_2, v_2^e\partial\lambda_1, v_2^e\partial\lambda_2 \mid 0 \le d < p, e \ge 0\}$, in Adams weight 2. Here, $|t^d\lambda_1\lambda_2| = 2p^2-2p-2d-2$, $|t^{dp}\lambda_1\lambda_2| = 2p^2-2p-2dp-2$, $|\partial\lambda_1| = 2p-2$, and $|\partial\lambda_2| = 2p^2-2$.
(4) $\mathbb{F}_p\{v_2^e\partial\lambda_1\lambda_2, e \ge 0\}$, in Adams weight 3 and degree $2p^2+2p-3$.

We argue that there are no non-trivial differentials. In a Bockstein spectral sequence, the differentials are trigraded. The differentials increase Adams weight by 1 and must increase v_2-degree by a positive amount. Differentials increase v_2-degree by at least 1, Adams weight by 1, and degree by -1. Any differential increasing v_2-degree by $b \geq 1$ maps $v_2^a x_1$ to $v_2^{a+2} x_2$ for some $a \geq 0$ and x_1, x_2 having no factor of v_2 where $\deg(v_2^{a+b} x_2) - \deg(v_2^a x_1) = -1$. This implies $\deg(x_2) - \deg(x_1) = b(2p^2 - 2) - 1$. Thus $b < 2$ since the maximum $\deg(x_2) - \deg(x_1)$ that can occur is $2p^2 - 2$. Since $\deg(x_2) - \deg(x_1) = 2p^2 - 1$ cannot occur, b cannot be 1, so we conclude that all differentials collapse. $\square$

The E_1-page of the v_2-Bockstein spectral sequence for $p = 2$



Now that we've stated the first observation, we state the second.

Proposition 10. *The syntomic cohomology calculation exhibits something that looks like Tate duality.*

Now that we've stated the first observation, we state the third.

Proposition 11. [HRW, Corollary 6.6.3] *For any prime $p \geq 2$ and type 3 p-local finite complex F, $F_*\mathrm{TC}(\ell)$ is finite.*

Proof. Let C denote the category of p-complete finite spectra V such that $V_*\mathrm{TC}(\ell)$ is finite. See that C is a thick subcategory of p-complete finite spectra: it contains 0 because $0_*\mathrm{TC}(\ell)$ is finite, it's closed under fibers and cofibers by observing the long exact sequence associated to a fiber or cofiber sequence, and retraction preserves tensoring with $\mathrm{TC}(\ell)$ thus will preserve finiteness.

By the Thick Subcategory Theorem, C must be $C_{\geq n}$ for some $n \geq 0$, the category of finite p-local spectra of type $\geq n$. Fix (i, j, k) so that the generalized Moore spectrum $V := \mathbb{S}/(p^i, v_1^j, v_2^k)$ of type 3. If we show that $V \in C_{\geq n}$, it follows that $k \leq 3$ in which case $C_{\geq 3} \subset C$, that is, the proposition would be proven.

We show that $V_*\mathrm{TC}(\ell)$ is finite for V the type 3 complex $\mathbb{S}/(p^i, v_1^j, v_2^k)$. Note that (i, j, k) have been picked so that there is a motivic spectral sequence with signature

$$\mathrm{gr}^{\star}_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^i, v_1^j, v_2^k) \Rightarrow V_*\mathrm{TC}(\ell)$$

.

Now we see $\mathrm{gr}^{\star}_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^i, v_1^j, v_2^k)$ may be resolved by finitely many copies of $\mathrm{gr}^{\star}_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p, v_1, v_2)$ in the following way. We inductively, for $1 \leq m < i$, exhibit the cofiber sequence

$$\mathrm{gr}^{\star}_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^m, v_1, v_2) \xrightarrow{p} \mathrm{gr}^{\star}_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^m, v_1, v_2) \to \mathrm{gr}^{\star}_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^{m+1}, v_1, v_2)$$

giving a cofiber sequence

$$\mathbb{S}/(p^m, v_1, v_2)_*\mathrm{TC}(\ell) \to \mathbb{S}/(p^m, v_1, v_2)_*\mathrm{TC}(\ell) \to \mathbb{S}/(p^{m+1}, v_1, v_2)_*\mathrm{TC}(\ell)$$

where the first two terms are finite by inductive hypothesis so that the final term is also finite. Repeating this process for the powers of v_1 and v_2 completes the proof. □

Finally we discuss the following corollary of the Proposition 11.

Corollary 12. [HRW, Theorem 6.6.4] *The Litchenbaum-Quillen conjecture holds for* $\mathrm{TC}(\ell)$*, that is,*

$$\mathrm{TC}(\ell)_{(p)} \to L_2^f \mathrm{TC}(\ell)_{(p)}$$

is a π_**-iso for* $* \gg 0$.

Proof. Recall that the L_2^f-localization map $\mathbb{S} \to L_2^f\mathbb{S}$ fits into a cofiber sequence

$$C \to \mathbb{S} \to L_2^f\mathbb{S}$$

Such that C is a filtered colimit of objects of $C_{\geq 3}$. In the last Proposition 11, we proved that the objects $C_{\geq 3}$ coincides with the spectra v such that $V_*\mathrm{TC}(\ell)$ is finite, thus C itself has the property that $C_*\mathrm{TC}(\ell)$ is finite. Applying $\mathrm{TC}(\ell)_*$ gives the fiber sequence

$$\mathrm{TC}(\ell) \to L_2^f\mathrm{TC}(\ell) \to \mathrm{TC}(\ell)_*V.$$

whose long exact sequence has $\mathrm{TC}(\ell) \to L_2^f\mathrm{TC}(\ell)$ must be an equivalence in degrees greater the highest degree appearing in $\mathrm{TC}(\ell)_*V$. □

References

[HRW] Jeremy Hahn, Arpon Raksit, and Dylan Wilson, *A motivic filtration on the topological cyclic homology of commutative ring spectra.*

[NS] Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, no. 2, 203–409.

Email address: prestoncranford2027@u.northwestern.edu