LECTURE 15: $K(1)$-LOCAL POWER OPERATIONS

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Abstract. This document is talk notes for Talk 15 in European Talbot 2025. We explain power operations in the $K(1)$-local setting and McClure's theorem.

Fix a prime p for the whole talk.

Power operations can be difficult to understand. It may be easier to study them in monochromatic settings such as in the $K(n)$-local or $T(n)$-local categories. In this talk we calculate the $K(1)$-local power operations. The calculation is known as McClure's theorem and a form of it is stated in Theorem 1.

Theorem 1 (McClure's Theorem, [BMMS])**.** *The homotopy ring of the p-completed free algebra $\mathrm{Free}^{\mathbf{E}_\infty}_{KU^\wedge_p}(x)^\wedge_p$ in $KU^\wedge_p$-local commutative rings on a class in degree 0 admits a δ-ring structure so that it is a free p-completed δ-ring on a class in degree 0.*

We explain why McClure's theorem calculates power operations in the $K(1)$-local category. From [Eur25, Talk 9] we know power operations in an $\mathcal{O}$-monoidal ∞-category Mod_R of modules over a spectrum R are calculated by the homotopy groups of the free algebra $\mathrm{Free}^{\mathcal{O}}_R(x)$ on a set with one element in degree 0. Thus the homotopy ring of p-completed free algebra $\mathrm{Free}^{\mathbf{E}_\infty}_{\mathrm{KU}^\wedge_p}(x)$ computes the power operations in the $\mathbf{E}_\infty$ monoidal category of $\mathrm{KU}^\wedge_p$-modules.

What the stated form of McClure's Theorem does is calculates power operations in the category of $\mathrm{KU}^\wedge_p$-modules as opposed to power operations in the $K(1)$-local category. The difference is a matter of p-Bockstein considerations $K(1)$-locally as explained later.

Now that we've explained how McClure's Theorem is used to compute power operations on $K(1)$-local spectra, we comment on history and literature on it in Remark 2.

Remark 2. *McClure's theorem is due to McClure in* [BMMS, Chapter IX]. *There is some discussion on the theorem by Hopkins in* [Hop] *and by Goerss-Hopkins in* [HG, Section 2.2].

In this talk we present a proof taught to the speaker by Ishan Levy, therefore what is good and correct should be attributed to Levy, and what is bad or incorrect should be attributed to the speaker.

We begin discussing a proof of McClure's theorem by discussing δ-rings. In [Eur25, Talk 9] we learned that a δ-ring is extra structure put on a commutative ring, and in the case of R being p-torsion free, it has the following two equivalent definitions: (1) the data of R and a map of sets $\delta : R \to R$ satisfying some equations or (2) the data of R and a ring homomorphism ψ lifting the Frobenius on R. Furthermore, for R p-torsion free, there is a bijection between choices of δ and lift $\psi(x) = x^p + p\delta(x)$ of the Frobenius map x^p. Now that we've recalled material about δ-rings, we discuss what the free p-complete δ-ring on a set with one element looks like in Proposition 3.

Proposition 3. *The free p-complete δ-ring on a set with one element is the commutative ring $\mathrm{Z}^\wedge_p[x_0, x_1, x_2, \dots]$ with δ-structure given by $\delta(x_i) = x_{i+1}$.*

Proof. Firstly we show that this is indeed a p-complete δ-ring: it is p-torsion free, so it suffices to see that the associated map φ with $\varphi(x_i) = x_i^p + px_{i+1}$ is a ring homomorphism that lifts the Frobenius. We've specified φ on the algebraically independent generators of the ring, so φ is a ring homomorphism. Also $x_i^p + px_{i+1}$ mod p is x_i^p so φ lifts Frobenius.

It remains to check the universal property of being free: we need to show that given a p-complete δ-ring R and a choice of $r \in R$ that there exists a unique map f of p-complete δ-rings $\mathrm{Z}_p^\wedge[x_0, x_1, x_2, \dots] \xrightarrow{f} R$ such that $f(x_0) = r$. Considering that $f(x_0) = r$, the relations $\delta(f(x_i)) = f(\delta(x_i)) = f(x_{i+1})$ ensure that the map $f(x_i) = \delta^i(r)$ makes f a map of p-complete δ-rings and is unique. $\square$

Now that we've discussed the free p-completed δ-ring, we will begin studying the p-completed free $\mathrm{KU}_p^\wedge$-algebra on a set with one element. Recall from [Eur25, Talk 9] we recall the following formula for the free object on a set with n elements in a $\mathcal{O}$-monoidal ∞-category of modules over a commutative ring spectrum R is

$$\mathrm{Free}_{\mathcal{C}}^{\mathcal{O}}(x_1, \dots, x_n) \cong \bigoplus_{n \geq 0} (\mathcal{O}(n) \otimes (S^0))^{\otimes_R n}_{h\Sigma_n}.$$

Applying this formula for $(x_1, \dots, x_n) = (x)$, $R = \mathrm{KU}_p^\wedge$, $\mathcal{O} = \mathbb{E}_\infty$ and p-completing gives us

$$\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x)_p^\wedge \cong \bigoplus_{n \geq 0} (\mathrm{KU}_p^\wedge \otimes B\Sigma_n)_p^\wedge$$

We will begin studying the free algebra in low n-ary parts. Firstly we recall some facts about group cohomology. In the following we take a group G and H a normal subgroup of G such that $[H : G] < \infty$. We also fix M a BG-module.

- There is a map $H^*(BG; M) \xrightarrow{res} H^*(BH; M)$ called the restriction given by pushforward.
- There is a map $H^*(BH; M) \xrightarrow{tr} H^*(BG; M)$ called the transfer given by viewing $BG \to BH$ as a finite covering map and sending a cocycle f to the map that sends a cell σ' to the sum of all $f(\sigma')$ for the cells σ' in the fiber over the cell σ.
- The double coset formula is the claim that $\mathrm{res} \circ \mathrm{tr}$ is the multiplicaiton by $[G : H]$ map on $H^*(BG; M)$.
- For $M = \mathbb{F}_p$ and $[G : H]$ is coprime to p there is an isomorphism onto the G-invariants

$$H_*(BG; \mathbb{F}_p) \cong H_*(BH; \mathbb{F}_p)_{G/H}.$$

  where G acts trivially on $\mathbb{F}_p$. The inclusion of G-invariants induces an inclusion

$$\begin{tikzcd}
H_*(BG; \mathbb{F}_p) \arrow[r, equal] \arrow[dr, hook] & H_*(BH; \mathbb{F}_p)_{G/H} \arrow[d, hook] \\
 & H_*(BH; \mathbb{F}_p).
\end{tikzcd}$$

Before we begin our calculations in low n-ary degrees, we remark that we will be performing group cohomology calculations in KU/p coefficients. We do this because KU/p is better behaved than $\mathrm{KU}_p^\wedge$: KU/p is "field-like" $K(1)$-locally. This is because $K(1)$ is the mod p Adams summand thus is a summand of KU/p. In particular, they share the same

Bousfield class, so $K(1)$-locally, KU/p enjoys the property that $K(1)$ has of being field-like in that modules over it are free as explained in [Eur25, Talk 5].

After computing the group cohomologies ${B\Sigma_n}^{KU/p}$, we will have to use this to calculate the group homologies $\mathrm{KU}^\wedge_p \otimes B\Sigma_n$. We do this in two steps. The first step is converting between cohomology and homology. Because KU/p is a field, (co)homology over it is free thus is determined by its rank over KU/p. It must be that cohomology and homology over KU/p give the same rank.

The second step is to convert from KU/p-homology to $\mathrm{KU}^\wedge_p$-homology. Due to a niceness condition ($KU/p \otimes B\Sigma_n$ being even), this conversion is doable. We record this in the following Lemma 4.

Lemma 4. *For X a space such that $KU/p \otimes B\Sigma_n$ is even, we have that*

$$KU^\wedge_p \otimes X \cong KU^\wedge_p \otimes_{KU/p} (KU/p \otimes X)$$

Proof. We consider the p-Bockstein sequence

$$0 \to \mathrm{KU}^\wedge_p \otimes X \xrightarrow{p} \mathrm{KU}^\wedge_p \otimes X \to \mathrm{KU}/p \otimes X \to 0$$

and its associated p-Bockstein spectral sequence with signature

$$H^p(\mathrm{KU}/p \otimes X; \pi_q(\mathrm{KU}^\wedge_p)) \Rightarrow H^{p+q}(\mathrm{KU}^\wedge_p \otimes X).$$

We assumed $KU/p \otimes B\Sigma_n$ is even. We also know $\pi_*(\mathrm{KU}^\wedge_p) \cong \mathrm{Z}^\wedge_p[\beta^\pm]$ with $\deg(\beta)$=2 so $\mathrm{KU}^\wedge_p$ is even. Thus the E_2-page is concentrated in even bidegrees, all differentials collapse, and the spectral sequence collapses to give the claim of the lemma. □

We've set up enough pre-requisities to compute $\mathrm{KU}^\wedge_p \otimes (B\Sigma_n)$ for $n < p$ in Proposition 5.

Proposition 5. *For $n < p$ we have $K(1)$-locally that*

$$KU^\wedge_p \otimes B\Sigma_n \cong KU^\wedge_p$$

so that

$$(KU^\wedge_p \otimes (B\Sigma_n))^\wedge_p \cong KU^\wedge_p\,.$$

Proof. We will proceed with our first transfer argument. Consider the trivial subgroup $* \subset \Sigma_n$. By the double coset formula, $\mathrm{res} \circ \mathrm{tr}$ is multiplication by $n!$ on $H^*(B\Sigma_k; KU/p)$. Since $n < p$, $n!$ is coprime to p, so $\mathrm{res} \circ \mathrm{tr}$ is the identity on $H^*(B\Sigma_k; KU/p)$. The composition $\mathrm{res} \circ \mathrm{tr}$ factors through $H^*(B*; KU/p) \cong KU/p$, therefore $H^*(B\Sigma_k; KU/p) \cong KU/p$. By Lemma 4, this implies the claimed statement. □

We've learned so far that there is only one power operation in n-ary parts $n < p$, and we've discussed nothing about a δ-ring structure. Recalling that δ-ring structure is a lift of Frobenius, the non-trivial data of a δ-structure to appear in n-ary degree $n = p$. Indeed, this will happen as observed in Proposition 6.

Proposition 6. *We have $K(1)$-locally that*

$$KU^\wedge_p \otimes B\Sigma_p \xrightarrow{\epsilon, \mathrm{tr}} KU^\wedge_p \oplus KU^\wedge_p$$

is an equivalence for some maps ϵ and tr *so that*

$$(KU_p^\wedge \otimes (B\Sigma_n))_p^\wedge \cong KU_p^\wedge \oplus KU_p^\wedge\,.$$

We will see that we can think of the δ-structure as coming from the transfer. To prove Proposition 6, we will perform transfer arguments. We have that $C_p \subset \Sigma_p$ is a maximal p-sylow subgroup, so we are interested in understanding $B\Sigma_n^{KU/p}$. This is calculated by the following Lemma 7.

Lemma 7. *We have that*

$$H^*(BC_p; KU/p) \cong \mathbb{F}_p[t]/(t^p) \otimes \mathbb{F}_p[\beta^\pm]$$

with $\deg(t) = 1$. *This imples* $B\Sigma_n^{KU/p}$ *is a rank* p *module over* KU/p.

Proof. To compute we use the Atiyah–Hirzebruch spectral sequence with signature

$$H^p(BC_p; \pi_q(KU/p)) \Rightarrow \pi_{p+q}H^*(BC_p; KU/p).$$

We recall that $H^*(BC_p; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda_{\mathbb{F}_p}(e)$ with $\deg(t) = 2$ and $\deg(e) = 1$. For degree reasons, the only possible non-trivial differential in the E_2 page is de. We will show $de = [p]_F(t)$ for F denoting the formal group law associated to KU/p so that

$$de = [p]_F(t) = v_1 t^p = \beta^{p-1} t^p$$

giving the claim. We consider the fiber sequence

$$BC_p \to BS^1 \xrightarrow{p} BS^1.$$

Continuing the sequence to the left gives us ΩBS^1 as the fiber of $BC_p \to BS^1$, and we know $\Omega BS^1 \cong S^1$, thus we have a spherical fibration

$$S^1 \to BC_p \to BS^1.$$

The associated Gysin sequence tells us that de is given by the Euler class. The Euler class is induced by pulling $t \in BS^1$ back by $BS^1 \xrightarrow{p} BS^1$ and thus is given by $[p]_F(t)$ as desired. □

From the above Lemma 7, the subgroup $C_p \subset \Sigma_p$ allows up to upperbound the rank of $B\Sigma_n^{KU/p}$ as being p. We will build upon the above calculation.

Now consider the subgroup $C_p \rtimes \mathrm{Aut}(C_p)$ in Σ_p. It has cardinality $p(p-1)$ and is once again a subgroup of index coprime to p. To make sense of $C_p \rtimes \mathrm{Aut}(C_p)$, we specify the action of $\mathrm{Aut}(C_p)$ on C_p, so we declare it to be the obvious one and not denote it in the notation of the semi-direct product.

Lemma 8. *We have*

$$H^*(B(C_p \rtimes \mathrm{Aut}(C_p)); KU/p) \cong \mathbb{F}_p[t^{p-1}]/((t^{p-1})^2) \otimes \mathbb{F}_p[\beta^\pm].$$

This implies $B(C_p \rtimes \mathrm{Aut}(C_p))^{KU/p}$ *is a rank* 2 *module over* KU/p.

Here are two more facts about group cohomology we recall now before the proof.

- For M a BG module we have that $H^0(BG; M)$ is the fixed points M^{hBG}.
- For $|G| < \infty$ and is invertible in M we have $H^*(BG; M) = 0$ for $* > 0$.

Now we state a proof of Lemma 8.

Proof. We seek to leverage our knowledge of $H^*(BC_p; KU/p)$, so we consider the exact sequence

$$0 \to BC_p \to B(C_p \rtimes \mathrm{Aut}(C_p)) \to B\,\mathrm{Aut}(C_p) \to 0$$

and it's associated Lyndon–Hochschild–Serre spectral sequence with signature

$$H^p(B\,\mathrm{Aut}(C_p); H^q((BC_p; KU/p)) \Rightarrow H^{p+q}(B(C_p \rtimes \mathrm{Aut}(C_p)); KU/p).$$

When $p = 2$, $\mathrm{Aut}(C_p)$ is trivial, and the desired statement is true. We assume $p > 2$ onwards.

Because $p > 2$, $\mathrm{Aut}(C_p)$ has magnitude coprime to p, thus

$$H^p(B\,\mathrm{Aut}(C_p); H^q((BC_p; KU/p))$$

vanishes when $p > 0$. We are therefore only interested in the line

$$E_2^{0,q} = H^0(B\,\mathrm{Aut}(C_p); H^q((BC_p; KU/p)) = H^q((BC_p; KU/p))^{hB\,\mathrm{Aut}(C_p)}.$$

There are no non-trivial differentials, to it remains only to study the fixed points calculation. The action of $\mathrm{Aut}(C_p)$ on C_p is given by exponentiation: $\mu^n \in \mathrm{Aut}(C_p) \cong C_{p-1}$ for μ a primitive $(p-1)$ root of unity sends t to $\mu^n t^n$. The fixed points are exactly when $n = 0$ or $n = p - 1$. □

Now that we've upper bounded rank 2, we have to show that $\mathrm{KU}_p^\wedge \otimes B\Sigma_p$ is not of rank 1, finally proving Proposition 6.

Proof. We recall from [Eur25, Talk 8] that the $K(n)$-local categories are ∞-semiadditive. This implies Tate vanishing for the $K(1)$-local category so that

$$\mathrm{KU}_{h\Sigma_p} \to \mathrm{KU}^{h\Sigma_p}$$

is an equivalence. The proof strategy going forward is to use Tate vanishing to distinguish two summands in $\mathrm{KU}_p^\wedge \otimes B\Sigma_p$ via the double-coset formula. We consider the ϵ map from KU/p to $\mathrm{KU}/p_{h\Sigma_p}$ given by taking homotopy orbits, and we denote Tr the transfer map $(KU/p)_{h\Sigma_p} \to \mathrm{KU}_p$. The diagram below depicts the setup.

$$\begin{tikzcd}
(KU/p)_{h\Sigma_p} \arrow[r, equal, "Nm"] \arrow[dr, "Tr"] & (KU/p)^{h\Sigma_p} \arrow[d, "can"] \\
KU/p \arrow[u, "\epsilon"] & KU/p
\end{tikzcd}$$

By an equivariant analogue of the double-coset formula, we have that the composition $\mathrm{Tr} \circ \epsilon$ is $p!$ which is 0 mod p. If $\mathrm{KU}_p^\wedge \otimes B\Sigma_p$ were of rank 1, then $\mathrm{Tr} \circ \epsilon$ would have been the identity, but it isn't, so $\mathrm{KU}_p^\wedge \otimes B\Sigma_p$ is not of rank 1. □

What we've shown in proving Proposition 6 the following equivalence:

$$\mathrm{KU}_p^\wedge \otimes B\Sigma_p \xrightarrow{\epsilon,\mathrm{tr}} \mathrm{KU}_p^\wedge \oplus \mathrm{KU}_p^\wedge$$

We call the power operation coming from ϵ by the name x and the power operation coming from transfer by $\delta(x)$.

We will begin discussing a δ-structure on the $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x)_p^\wedge$. We will do so by stating an endomorphism ψ of sets on $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x)_p^\wedge$ and show that ψ is indeed a ring homomorphism.

A map of sets ψ is supposed to be a lift of Frobenius, thus we will find it by picking a power operation in the p-ary degree. We declare it to be $(1,0)$ in $\mathrm{KU}_p^\wedge \otimes B\Sigma_p \xrightarrow{\epsilon,\mathrm{tr}} \mathrm{KU}_p^\wedge \oplus \mathrm{KU}_p^\wedge$. In other words, we pick the operation such that $\epsilon(\psi) = 1$ and $\mathrm{tr}(\psi) = 0$. We need to show it preserves addition and multiplication.

Remark 9. *One may wonder about the approach of instead showing the map of sets δ satisfies the two relations to show that it gives a δ-ring structure. This requires understanding $\epsilon(\delta(x))$ and $\mathrm{tr}(\delta(x))$. To know $\epsilon(\delta(x))$, we look at the equation*

$$\psi(x) = x^p + p\delta(x)$$

and apply ϵ to learn

$$1 = \epsilon(x^p) + p\epsilon(\delta(x)).$$

By definition of ϵ, $\epsilon(x^p) = 1$, so solving for $\epsilon(\delta(x))$ gives $\epsilon(\delta(x)) = 0$. To understand $\mathrm{tr}(\delta(x))$, we now apply apply tr to

$$\psi(x) = x^p + p\delta(x)$$

to get

$$0 = \mathrm{tr}(x^p) + p\,\mathrm{tr}(\delta(x)).$$

A double-coset argument shows $\mathrm{tr}(x^p) = p!$, thus solving for $\mathrm{tr}(\delta(x))$ gives $\mathrm{tr}(\delta(x)) = -\frac{1}{(p-1)!}$.

Firstly we will show that for $x, y \in \pi_*(\mathrm{Free}_{\mathrm{KU}_p^\wedge}(z))$ that $\psi(x) + \psi(y) = \psi(x+y)$. To do this, we will need to understand the free algebra $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$ generated by two elements.

From the formula we've seen before, we have that

$$\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y) \cong \bigoplus_{n\geq 0} (\mathrm{KU}_p^\wedge \otimes \mathrm{KU}_p^\wedge)_{h\Sigma_n}^{\otimes_{\mathrm{KU}_p^\wedge} n}$$

The relation we seek to show is in p-ary degree of $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$, so we study more generally what the n-ary part of $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$ looks like in the following Lemma 10

Lemma 10.

$$(KU_p^\wedge \otimes KU_p^\wedge)_{h\Sigma_n}^{\otimes_{KU_p^\wedge} n} \cong \bigoplus_{n=0}^{n} KU_p^\wedge \otimes B(\Sigma_i \times \Sigma_{p-i})$$

Proof. First we must calculate the homotopy orbits of $(\mathrm{KU}_p^\wedge \otimes \mathrm{KU}_p^\wedge)^{\otimes_{\mathrm{KU}_p^\wedge} n}$. The action of Σ_n permutes the p pairs $\mathrm{KU}_p^\wedge \otimes \mathrm{KU}_p^\wedge$. For convenience we label the first and second copies x and y respectively. On the level of sets, the set being acted upon looks like the set

$$\mathrm{KU}_p^\wedge\{x,y\}^n$$

with 2^n elements. An element of the set looks like $\mathrm{KU}_p^\wedge(x,x,y,x,y,\dots)$, a sequence of n elements of $\{x,y\}$. The action of Σ_p preserves the number of x's in a sequence, so there must be at least $n+1$ orbits corresponding to the possible number $(0,1,\dots,n-1,n)$ number of

x's in a sequence. Since Σ_p acts transitively any subset of sequences with a fixed amount of x's, this concludes our calculation of the orbits.

It remains to justify the stabilizer calculation. On an orbit of sequences with i $x's$ and $n-i$ $y's$, the stabilizer on the $x's$ is given by Σ_i and the stabilizer on the $y's$ is given by Σ_{n-i}. □

Now that we have some handle on the p-ary part of the free algebra on two elements, we can describe what we want. By the universal property of free algebras, to pick the element $x+y$ in $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$ is the same as picking a map

$$\mathrm{KU}_p^\wedge(z) \to \mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y).$$

We define $\psi(x)+\psi(y) \in \mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$ to be the image of $\psi(x)+\psi(y) \in \mathrm{Free}_{\mathrm{KU}_p^\wedge}(z)$ by the above inclusion. At last, we now have description of $\psi(x)+\psi(y)$ and $\psi(x+y)$ as power operations living in the p-ary part of $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$. We begin.

Lemma 11. *We have that* $\psi(x)+\psi(y) = \psi(x+y)$ *in* $\mathrm{Free}_{KU_p^\wedge}(x,y)$.

Proof. From the prior Lemma 10 we see that the p-ary part of $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$ is given by

$$x^p\,\mathrm{KU}_p^\wedge \otimes B(\Sigma_p) \oplus \bigoplus_{i=1}^{p-1} x^i y^{p-i}\,\mathrm{KU}_p^\wedge \otimes B(\Sigma_i \times \Sigma_{p-i}) \oplus y^p\,\mathrm{KU}_p^\wedge \otimes B(\Sigma_p).$$

Firstly we show that $\psi(x)+\psi(y) = \psi(x+y)$ after applying transfer and look at the middle terms. Consider the trivial subgroup $* \subset \Sigma_i \times \Sigma_{p-i}$. Since $|\Sigma_i \times \Sigma_{p-i}| = i!(p-1)!$ is coprime to p, the double coset formula once again tells us that

$$x^i y^{p-i}\,\mathrm{KU}/p \otimes B(\Sigma_i \times \Sigma_{p-i}) \xrightarrow{Tr} x^i y^{p-i}\,\mathrm{KU}/p \otimes B* \cong x^i y^{p-i}\,\mathrm{KU}/p$$

is an equivalence. This shows that $\mathrm{tr}(\psi(x+y)$ must be 0 on the mixed terms. Since $\mathrm{tr}(\psi) = 0$, we have that

$$\mathrm{tr}(\psi(x)+\psi(y)) = \mathrm{tr}(\psi(x)) + \mathrm{tr}(\psi(y)) = 0+0 = 0$$

It remains to look at the unmixed terms given by x^p and y^p. We will study these one at a time beginning with the x^p term. We consider the map

$$\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y) \to \mathrm{Free}_{\mathrm{KU}_p^\wedge}(z)$$

given by mapping the elements x to z and y to 0. The induced map on p-ary parts gives a projection of the ϵ map which we call ϵ_x. We see $\epsilon_x(x) = 1$ and $\epsilon_x(y) = 0$. By symmetry we can swap x, y and rerun the argument. Finally we have $\epsilon = \epsilon_x + \epsilon_y$ which shows $\epsilon(\psi(x)+\psi(y)) = 1$ as desired. □

Now that we've shown additivity in Lemma 11, we show multiplicativity. The power operations $\psi(xy)$ and $\psi(x)\psi(y)$ live in $2p$-ary degree. We see x and y live in degree 1 so xy lives in degree 2, and applying ψ makes it live in degree $2p$. Likewise the operations $\psi(x)$ and $\psi(y)$ live in degree p each, and their product lives in degree $2p$.

Lemma 12. *We have that* $\psi(x)\psi(y) = \psi(xy)$ *in* $\mathrm{Free}_{KU_p^\wedge}(x,y)$.

Proof. The operations $\psi(x)\psi(y) = \psi(xy)$ are both of (x, y)-degree (p, p), so that is the only part of the $2p$-ary degree part of $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x, y)$ we need to look at. That is

$$x^p y^p \,\mathrm{Free}_{\mathrm{KU}_p^\wedge} \otimes B(\Sigma_p \times \Sigma_p).$$

One can go through a very similar argument before, by considering the subgroup

$$(C_p \rtimes \mathrm{Aut}(C_p)) \times C_p \rtimes \mathrm{Aut}(C_p) \subset \Sigma_p \times \Sigma_p$$

and using that KU/p has a Kunneth formula, we get the following equivalence:

$$\mathrm{KU}_p^\wedge \otimes B(\Sigma_p \times \Sigma_p) \xrightarrow{\epsilon\otimes\epsilon, \mathrm{tr}\otimes\epsilon, \epsilon\otimes\mathrm{tr}, \mathrm{tr}\otimes\mathrm{tr}} (\mathrm{KU}_p^\wedge)^{\oplus 4}.$$

As before $\mathrm{tr}(\psi) = 0$ makes it so that we only need to see that $\psi(x)\psi(y) = \psi(xy)$ on the $\epsilon \otimes \epsilon$ part.

To study the $\epsilon \otimes \epsilon$ part, we again project onto say the x part first by considering the map

$$\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x, y) \to \mathrm{Free}_{\mathrm{KU}_p^\wedge}(z)$$

induced by setting $x \to z$ and $y \to 0$. This again gives a $(\epsilon \otimes \epsilon)_x$ component that is 1 on x and 0 on y. We get the same things with x, y swapped, and we get the decomposition

$$\epsilon \otimes \epsilon = (\epsilon \otimes \epsilon)_x + (\epsilon \otimes \epsilon)_y$$

again giving $\psi(x)\psi(y) = \psi(xy)$ as desired. $\square$

Now that we've proven Lemma 11 and Lemma 12, we conclude that ψ is a ring homomorphism so that we have finished constructing a δ-ring structure on $\pi_*(\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x))$.

It only remains to exhibit its universal property: that as a p-complete δ-ring, it's free on one object. Firstly we consider the map

$$\mathrm{Free}_\delta(x)_p^\wedge \to \mathrm{Free}_{\mathrm{KU}_p^\wedge}(y)_p^\wedge$$

given by sending x to y. We can do this because $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(y)_p^\wedge$ is considered a p-complete δ-ring and we use the universal property of $\mathrm{Free}_\delta(x)_p^\wedge$. To show this is an equivalence, it suffices to show that this map of δ-rings induces an isomorphism of underlying rings. We do this by showing the underlying map of rings is a surjection and an injection of sets.

We begin by showing the surjectivity. What we do is show that it is surjective onto each n-ary part. Note that we've already done this for the n-ary degrees $n \leq p$: for $n < p$, it is surjected onto by $\mathrm{Z}_p^\wedge\{x^n\}$, and in p-ary degree it is surjected onto by $\mathrm{Z}_p^\wedge\{x^p, \delta(x)\}$.

Before we prove the surjectivity claim, we prove it in $2p$-ary degree in Example 13 as an easier to understand case than the general proof.

Example 13. *We show*

$$\mathrm{Free}_\delta(x) \to \mathrm{Free}_{KU_p^\wedge}(y)_p^\wedge$$

surjects onto the $2p$-ary part.

Proof. The $2p$-ary part of $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(y)_p^\wedge$ is given by

$$(\mathrm{KU}_p^\wedge(B\Sigma_{2p}))_p^\wedge.$$

Consider the following subgroup

$$((C_p \rtimes \mathrm{Aut}(C_p)) \times (C_p \rtimes \mathrm{Aut}(C_p))) \rtimes C_2 \subset \Sigma_{2p}$$

where the outermost semidirect product with C_2 acts by permuting the two factors on either side of the single direct product taken.

Note that the maximum power of p appearing in $|\Sigma_{2p}| = (2p)!$ is $p^3 = 8$ when $p = 2$ and is p^2 when p^2, thus the subgroup constructed has index in Σ_{2p} coprime to p. This implies that there is an injection

$$H^*(B\Sigma_{2p}, \mathrm{KU}/p) \to H^*(B\left((C_p \rtimes \mathrm{Aut}(C_p)) \times (C_p \rtimes \mathrm{Aut}(C_p))\right) \rtimes C_2; \mathrm{KU}/p)$$

so it suffices to show that the composition

$$\mathrm{Free}_\delta(x) \to \mathrm{KU}_p^\wedge \otimes B\Sigma_{2p} \to \mathrm{KU}_p^\wedge \otimes B\left((C_p \rtimes \mathrm{Aut}(C_p)) \times (C_p \rtimes \mathrm{Aut}(C_p))\right) \rtimes C_2$$

is surjective.

We've seen that the cohomology of one of the semi-direct products has rank 2, so taking the direct product gives rank 4 by the Kunneth formula for KU/p. Finally, we've also seen that the outermost semidirect produt gives fixed points, and this results in the identification of $x^p\delta(x)$ and $\delta(x)x^p$, giving rank 3. This is surjected onto to by x^{2p}, $x^p\delta(x)$, and $\delta(x)^2$. ☐

Now that we've seen Example 13, we now move onto the full Proposition 14.

Proposition 14. *We have*

$$\mathrm{Free}_\delta(x) \to \mathrm{Free}_{KU_p^\wedge}(y)_p^\wedge$$

is surjective.

Proof. We show this at each n-ary part. We recall how to construct a p-Sylow subgroup of Σ_n. Write n in base p as $\sum_{i=0}^k a_i p^i$ with $0 \leq a_i < p$ and k minimal. A p-Sylow subgroup is given by

$$\prod_{i=0}^k \prod_{j=0}^{a_i} \rtimes_{m=0}^{a_i} C_p.$$

This implies that

$$\prod_{i=0}^k \prod_{j=0}^{a_i} (\rtimes_{m=0}^{a_i} (C_p \rtimes \mathrm{Aut}(C_p)) \rtimes C_{a_i})$$

is a subgroup of Σ_n of index coprime to p. We once again proceed with the strategy of showing

$$\begin{aligned} \mathrm{Free}_\delta(x) &\to \mathrm{KU}_p^\wedge \otimes B\Sigma_n \\ &\to \mathrm{KU}_p^\wedge \otimes B \prod_{i=0}^k \prod_{j=0}^{a_i} (\rtimes_{m=0}^{a_i} (C_p \rtimes \mathrm{Aut}(C_p)) \rtimes C_{a_i}) \end{aligned}$$

is surjective. The idea is that each

$$\rtimes_{m=0}^{a_i} (C_p \rtimes \mathrm{Aut}(C_p)$$

corresponds in group cohomology with all the words of length a_i with letters x^{p^k} and $\delta^k(x)$, and then the outermost semi-direct producting with C_{a_i} in

$$(\rtimes_{m=0}^{a_i}(C_p \rtimes \mathrm{Aut}(C_p)) \rtimes C_{a_i})$$

ensures, in group cohomology, that the words

$$x^{p^{k^a}}\delta^{k^{a_i-a}}(x), \delta^k(x)x^{p^{k^a}}\delta^{k^{a_i-a-1}}(x), \dots, \delta^{k^{a_i-a}}(x)x^{p^{k^a}}$$

are identified. We see that everything is expressible in terms of x and applying δ which shows surjectivity.

$\square$

It remain to argue injectivity. The idea is to map both free rings into some δ-ring R with a very large homotopy ring, but we also want it's δ-ring structure to be well behaved, so we will need to introduce some facts about Witt vectors as we will use them to build R.

- A commutative ring A of characteristic p is said to be perfect if the Frobenius $x \to x^p$ is an isomorphism on R.
- To A one can associate the Witt vector ring $W(A)$. There are several ways to define this, and we state none of them here.
- When A is perfect, characteristic p, and p-torsion free, we have that $W(A)/p = A$.
- For A perfect, characteristic p, and p-torsion free, Witt vectors $W(A)$ have a unique δ-ring structure coming from the fact that there is a unique map $W(A)/p = A \to W(A)/p = A$, the Frobenius.
- The δ-ring structure in $W(A)$ has the following universal property: for B a δ-ring and $f : B \to A$ a map of rings, there exists a unique δ-ring map $B \to W(A)$ making the following diagram of ring maps commute:

$$\begin{tikzcd} & W(A) \arrow[d, "\pi"] \\ B \arrow[ur, dashed, "\exists!"] \arrow[r, "f"] & A \end{tikzcd}$$

.

Now we are prepared to show injectivity in Propositon 15.

Proposition 15. *The ring map*

$$\mathrm{Free}_\delta(x)^\wedge_p \to \mathrm{Free}_{KU^\wedge_p}(y)^\wedge_p$$

is injective.

Proof. As said before, idea is to form a commutative $\mathrm{KU}^\wedge_p$-local ring R with large π_* and a well-behaved δ-ring structure that fits into the diagram

$$\mathrm{Free}_\delta(x) \to \mathrm{Free}_{\mathrm{KU}^\wedge_p}(y)^\wedge_p \to R.$$

We define

$$R := (\Sigma^\infty_+ \mathbb{N}[\frac{1}{p}])^{\oplus\infty} \otimes \mathrm{KU}^\wedge_p\,.$$

This gives a commutative $\mathrm{KU}_p^\wedge$-algebra because the monoidal multiplication on $\mathbb{N}$ is commutative and we've tensored with $\mathrm{KU}_p^\wedge$. We've also performed adjoining of $\frac{1}{p}$, which we recall to be

$$\mathbb{N}[\frac{1}{p}] = \mathrm{colim}(\mathbb{N} \xrightarrow{p} \mathbb{N} \xrightarrow{p} \mathbb{N} \xrightarrow{p} \cdots).$$

We see $\pi_* R$ is perfect as

$$\pi_* R \cong \mathrm{KU}_{p,*}^\wedge[x_0^{\frac{1}{p^\infty}}, x_1^{\frac{1}{p^\infty}}, \dots]$$

so that

$$W(\pi_*(R))/p \cong \pi_*(R).$$

We begin constructing a map $\mathrm{Free}_\delta(x)_p^\wedge \to R$. Consider the inclusion

$$\mathrm{Free}_\delta(x)_p^\wedge \to \mathrm{Free}_\delta(x)[\psi^{-1}]_p^\wedge$$

and note that

$$\pi_*(\mathrm{Free}_\delta(x)[\psi^{-1}]_p^\wedge/p) \cong \mathrm{Z}_p^\wedge[x_0^{\frac{1}{p^\infty}}, x_1^{\frac{1}{p^\infty}}, \dots]$$

is perfect. Thus we have

$$W(\mathrm{Free}_\delta(x)[\psi^{-1}]_p^\wedge)/p \cong \mathrm{Free}_\delta(x)[\psi^{-1}]_p^\wedge.$$

By the universal property of Witt vectors, associated to the ring map

$$\mathrm{Free}_\delta(x) \to \pi_*(R)$$

is a unique δ-ring map making the following diagram commute

$$\begin{tikzcd}
 & W(\pi_*(R)) \arrow[d, "\pi"] \\
\mathrm{Free}_\delta(x) \arrow[ur, dashed, "\exists!"] \arrow[r, "f"] & \pi_*(R).
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
 & R \\
\mathrm{Free}_\delta(x)_p^\wedge \arrow[ur] \arrow[r, "f"] & \mathrm{Free}_{\mathrm{KU}_p^\wedge}(y)_p^\wedge \arrow[u]
\end{tikzcd}$$

commutes because it is induced by the following diagram of set maps:

$$\begin{tikzcd}
 & x_0 \in \pi_* R \\
\{x\} \subset \mathrm{Free}_\delta(x)_p^\wedge \arrow[ur] \arrow[r, "f"] & \{y\} \subset \mathrm{Free}_{\mathrm{KU}_p^\wedge}(y)_p^\wedge. \arrow[u]
\end{tikzcd}$$

Finally, since $\mathrm{Free}_\delta(x)^\wedge_p \to \mathrm{Free}_\delta(x)[\psi^{-1}]^\wedge_p$ is injective, we have that $\mathrm{Free}_\delta(x)^\wedge_p \to R$ is injective. We also know that $\mathrm{Free}_{\mathrm{KU}^\wedge_p}(y)^\wedge_p \to R$ is injective on the level of homotopy groups as y^n goes to y_{n-1} and applying δ's changes p-adic valuation. $\square$

Proposition 14 and Proposition 15 together show that the map $\mathrm{Free}_\delta(x)^\wedge_p \to \mathrm{Free}_{\mathrm{KU}^\wedge_p}(y)^\wedge_p$ is an isomorphism of p-complete δ-rings on the level of homotopy groups. This concludes the proof of Theorem 1.

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