

GENERALIZED THOM SPECTRA, CONJUGATE STACKS, AND TRUNCATED BROWN-PETERSON SPECTRA

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ABSTRACT. We propose a definition conjugate stack for a class of suitable $\mathbb{E}_3$ -MU-algebras that includes the truncated Brown-Peterson spectra $BP\langle n \rangle$ for all heights n . Then we calculate $BP\langle n \rangle^{\text{conj}}$ as a quotient stack in terms of Witt vectors. To perform this computation we study generalized Thom spectra and their relative Topological Hochschild Homology. We exhibit the Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$ as a $\mathbb{E}_3$ -MU-Thom spectrum, the p -local Eilenberg-Mac Lane spectra $H\mathbb{Z}/p^n$ (except for $H\mathbb{Z}/4$) and $H\mathbb{Z}_{(p)}$ as $\mathbb{E}_3$ - $\text{MU}_{(p)}$ -Thom spectra, and the connective complex K-theory spectrum ku as $\mathbb{E}_1$ -MU-Thom spectra. We accompany these “go” results with “no-go” results that show these “go” results do not admit $\mathbb{E}_\infty$ -refinements.

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1. INTRODUCTION

The areas of math p -adic arithmetic geomtry and homotopy theory have have experienced rapid development in the last decade. This time of rapid development begins with [BMS19] where the Bhatt-Morrow-Szhole study the (p -completed) Topological Hochschild Homology $\text{THH}(R)_p^\wedge$ of suitable discrete commutative rings known as quasisyntomic rings. Furthermore the authors study a filtration on $\text{THH}(R)_p^\wedge$ known as the motivic filtration $\text{fil}_{\text{mot}}^* \text{THH}(R)_p^\wedge$. Topologiical Hochschild Homology has a property known as being *cyclotomic* that allows Bhatt-Morrow-Szhole to also study variations of the motivic filtration $\text{fil}_{\text{mot}}^* \text{THH}(R)_p^\wedge$ known as (Nygaard-completed) *prismatic cohomology* and *syntomic cohomology*.

Later, in [Bha22], [BL22], [Dri24], Bhatt-Scholze, Bhatt-Lurie, and Drinfeld introduce “stacky” approaches to the motivic filtration $\text{fil}_{\text{mot}}^* \text{THH}(R)_p^\wedge$ known as prismaticization. For example, to a p -complete discrete quasisyntomic ring R , there exists a prismaticization stack $(\text{Spf}(R)^\wedge)_{s=0}$ known as the arithmetic conjugate stack of R that geometrizes the motivic filtration $\text{fil}_{\text{mot}}^* \text{THH}(R)_p^\wedge$ in the sense that

$$\text{gr}_{\text{mot}}^t \text{THH}(R)_p^\wedge \simeq R\Gamma((\text{Spf}(R)^\wedge)_{s=0}, \mathcal{O}\{t\})[2t]$$

as graded commuative rings.

On the side of homotopy theory, in [HW22], Hahn-Raksit-Wilson study a filtration $\mathrm{fil}_{\mathbb{E}_\infty - \mathrm{ev}}^* \mathrm{THH}(R)$ on Topological Hochschild of suitable commutative ring spectra known as chromatically quasisyntomic ring spectra. Later, in [Pst23], Pstragowski studies a filtration $\mathrm{fil}_{\mathrm{ev}}^* \mathrm{THH}(R)$ on Topological Hochschild of suitable associative ring spectra known as $\mathbb{E}_2$ -ring spectra.

The study “stacky” approaches to motivic filtrations for ring spectra, which can be reasonably called the study of motivic filtrations for spectral schemes, is under investigation in the math community. In [Gre25], Gregoric studies a notion of spectral stacks of commutative ring spectra, called even periodization, that recovers the motivic filtration $\mathrm{fil}_{\mathbb{E}_\infty - \mathrm{ev}}^* \mathrm{THH}(R)$ of commutative ring spectra R . In his thesis [Dev25, Chapters 2 and 7], Devalapurkar describes ongoing work (with collaborators) on the following topics: the study even stacks of commutative ring spectra, a “stacky” version of the even filtration of commutative ring spectra, the study of “decompleted cycltomic spectra” (in the Nygaard-completion sense), and the study of spectral prismatization stacks of commutative ring spectra. One such example of a spectral prismatization stacks of commutative ring spectra includes a proposed definition of the conjugate stack associated to a suitable commutative ring spectrm.

In this article we propose a definition of conjugate stack R^{conj} associated to certain ring spectra of a class of (mildly commutative) associative ring spectra that includes the truncated Brown-Peterson spectra $\mathrm{BP}\langle n \rangle$ at all heights n . We list three points that defend our proposed definition.

- (1) Recall that given p -complete discrete quasisyntomic ring R , there exists its associated conjugate $(\mathrm{Spf}(R)^\mathcal{N})_{s=0}$ that geometrizes the motivic filtration $\mathrm{fil}_{\mathrm{mot}}^* \mathrm{THH}(R)_p^\wedge$ in the sense that there exists an equivalence

$$\mathrm{gr}_{\mathrm{mot}}^t \mathrm{THH}(R)_p^\wedge \cong R\Gamma((\mathrm{Spf}(R)^\mathcal{N})_{s=0}, \mathcal{O}\{t\}) [2t]$$

as graded commutative rings. Likewise, for R a suitable ring spectrum so that R^{conj} is defined, R^{conj} geomtetrizes the even filtration $\mathrm{fil}_{\mathrm{ev}}^* R$ in the sense that there exists an equivalence

$$\mathrm{gr}_{\mathrm{ev}}^t \mathrm{THH}(R) \simeq R\Gamma(R^{\mathrm{conj}}, \mathcal{O}\{t\}) [2t]$$

as spectra for each t .

- (2) For R a p -complete discrete quasisyntomic ring R that is suitable as a ring spectrum so that both $(\mathrm{Spf}(R)^\mathcal{N})_{s=0}$ and R^{conj} are defined, the two stacks agree. In paticular, there exists an equivalence

$$\widehat{R^{\mathrm{conj}}}_p \cong (\mathrm{Spf}(R)^\mathcal{N})_{s=0}$$

between formal p -competion $\widehat{R^{\mathrm{conj}}}_p$ of R^{conj} and the arithmetic conjugate stack $(\mathrm{Spf}(R)^\mathcal{M})_{s=0}$.

- (3) For R a stuitable commutative ring spectrum so that R^{conj} and Devalapurkar’s proposed definition in [Dev25, Chapter 7] of conjugate stack associated to R are defined, the two definitions coincide. Indeed, whereas Devalapurkar defines an even stack $\mathrm{Spev}(R)$ associated to R and defines the conjugate stack in terms of even stacks, one gets that Devalapurkar’s definition gives that the conjugate stack associated to R is $\mathrm{Spev}(\mathrm{THH}(R))$ itself, and one can show that $\mathbb{R}^{\mathrm{conj}} \simeq \mathrm{Spev}(\mathrm{THH}(R))$ as graded stacks.

We introduce our definition of conjugate stack as well as our Main Theorem B.

Definition 1.1. Let R be a connective $\mathbb{E}_3$ -MU-algebra such that $\mathrm{THH}(R/\mathrm{MU})$ is even, and define conjugate stack R^{conj} of R denote the graded quotient stack associated to the graded groupoid scheme

$$\mathrm{Spec}(\mathrm{THH}_{2*}(R/\mathrm{MU})) \xleftrightarrow[\eta_R]{\eta_L} \mathrm{Spec}(\mathrm{THH}_{2*}(R/\mathrm{MU}^{\otimes 2})).$$

where η_L and η_R denote the maps induced by the left and right units of the Hopf algebroid $(\mathrm{MU}, \mathrm{MU} \otimes \mathrm{MU})$.

Notation 1.2. We establish notation to state Theorem B. We let W denote the p -typical Witt vector group scheme, F denote Frobenius, and $W[F^n]$ denote the kernel of F^n group scheme.

Theorem B. (i) Let R be a connective $\mathbb{E}_3$ -MU-algebra such that $\mathrm{THH}(R/\mathrm{MU})$ is even. Then R^{conj} satisfies the following properties. Firstly, there is an equivalence of spectra

$$\mathrm{gr}_{\mathrm{ev}}^t \mathrm{THH}(R) \simeq R\Gamma(R^{\mathrm{conj}}, \mathcal{O}\{t\})[2t]$$

for $\mathrm{fil}_{\mathrm{ev}}^*$ denoting Pstragowski's even filtration [Pst23, Theorem 1.3]. Furthermore is a strongly convergent multiplicative spectral sequence of signature

$$H^s(R^{\mathrm{conj}}, \mathcal{O}\{t\}) \Rightarrow \pi_{2t-s} \mathrm{THH}(R).$$

Lastly, if R is furthermore a discrete p -complete quasisyntomic commutative ring such that $\mathrm{THH}(R/\mathrm{MU})$ has bounded p -power torsion then there is an isomorphism of fqc stacks

$$\widehat{R^{\mathrm{conj}}}_p \cong (\mathrm{Spf}(R)^{\mathcal{N}})_{s=0}$$

between the formal p -completion of R^{conj} and the zero fiber $(\mathrm{Spf}(R)^{\mathcal{M}})_{s=0}$ of the Nygaard filtered prismaticization of $\mathrm{Spf}(R)$, the familiar definition of conjugate stack of R .

(ii) Let $\mathrm{BP}\langle n \rangle_{p=\dots=v_n=0}^{\mathrm{conj}}$ denote the mod $(p, v_1, \dots, v_n)$ fiber

$$\mathrm{BP}\langle n \rangle^{\mathrm{conj}} \times_{\mathrm{Spec}(\mathbb{Z}_p[v_1, \dots, v_n])/\mathbb{G}_m} B\mathbb{G}_{m, \mathbb{F}_p}$$

of $\mathrm{BP}\langle n \rangle^{\mathrm{conj}}$. We have

$$\mathrm{BP}\langle n \rangle_{p=\dots=v_n=0}^{\mathrm{conj}} = [F_*^{m+1} W / F_* W] / \mathbb{G}_m = [\mathbb{G}_a / W[F^n]] / \mathbb{G}_m$$

as affine group schemes over $\mathrm{Spec}(\mathbb{Z}_p[v_1, \dots, v_n])/\mathbb{G}_m$ where the first quotient is associated to $pF^n : F_* W_{\mathbb{F}_p} \rightarrow F_*^{n+1} W_{\mathbb{F}_p}$ and the second quotient is associated to the shift by $w_0 : W[F^n] \rightarrow \mathbb{G}_a$.

Corollary 1.3 (Corollary 4.18). We have

$$[\mathrm{Spec}(\mathrm{THH}_*(\mathbb{F}_p))/\mathbb{G}_m] \cong \mathbb{F}_p^{\mathrm{conj}} \cong \mathbb{G}_a / \mathbb{G}_m$$

as stacks over $\mathrm{Spec}(\mathbb{F}_p)/\mathbb{G}_m$ and that

$$\bigoplus_{t \in \mathbb{Z}} R\Gamma(\mathbb{F}_p^{\mathrm{conj}}, \mathcal{O}\{t\})[2t] \cong \mathbb{F}_p[x],$$

with $|x| = 2$, as graded $\mathbb{F}_p$ -modules. Thus the fact that $\mathbb{F}_p^{\mathrm{conj}} \cong \mathbb{G}_a / \mathbb{G}_m$ as stacks over $\mathrm{Spec}(\mathbb{F}_p)/\mathbb{G}_m$ is a "stacky" version of Bökstedt periodicity.

Corollary 1.4 (Corollary 4.23). *We have*

$$\mathrm{Spec}(\mathrm{THH}_*(\mathrm{BP}\langle n \rangle)/(p, \dots, v_n)) \cong \mathrm{BP}\langle n \rangle_{p=v_1=\dots=v_n=0}^{\mathrm{conj}} \cong [F_*^{n+1}W/F_*W]$$

as graded affine group schemes over $\mathrm{Spec}(\mathbb{F}_p)$ and that

$$\bigoplus_{t \in \mathbb{Z}} R\Gamma(\mathrm{BP}_{p=v_1=\dots=v_n=0}^{\mathrm{conj}}, \mathcal{O}\{t\})[2t] \cong \mathbb{F}_p[\mu_{n+1}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_1, \dots, \lambda_{n+1}),$$

with $|\mu_{n+1}| = 2p^{n+1}$ and $|\lambda_i| = 2p^i - 1$, as graded $\mathbb{F}_p$ -algebras. This is a “stacky” version of the calculation

$$\pi_*(\mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{BP}\langle n \rangle))/(p, v_1, \dots, v_n) \cong \mathbb{F}_p[\mu_{n+1}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_1, \dots, \lambda_{n+1})$$

of [Ang26].

To study conjugate stacks R^{conj} of suitable ring spectra R , we first develop techniques for understanding $\mathrm{THH}_*(R/\mathrm{MU})$ as a R_* -Hopf algebra. CITE A LOT OF STUFF.

Consider Bökstedt periodicity [Bök85], the fundamental calculation that $\mathrm{THH}_*(\mathbb{F}_p) \simeq H_*(\Omega S^3; \mathbb{F}_p)$ as graded (discrete) rings. The more modern and topological way to prove Bökstedt periodicity requires understanding $\mathbb{F}_p$ as a Thom spectrum. The expression of $\mathbb{F}_p$ as a Thom spectrum is known as the Hopkins-Mahowald Theorem (Theorem ??). The $p = 2$ case is due to Mahowald in [Mah79], and the $p > 2$ case is due to Mike Hopkins.

Our strategy to calculate $\mathrm{THH}_*(R/\mathrm{MU})$ (for suitable R) as a R_* -Hopf algebra (to prove Theorem B) is summarized following way. Will be to firstly express $\mathbb{Z}/p^n$ (but not for $\mathbb{Z}/4$) and $\mathbb{Z}_{(p)}$ as multiplicatively coherent generalized Thom spectra in $\mathrm{Mod}_{\mathrm{MU}}$ as stated in the following Theorem A. Secondly, we study a “THH-of-Thom spectra” formula that accounts not just for algebraic structure but also augmented and coalgebraic structure provided by the geometry of generalized Thom spectra as stated in the following Proposition ?? . Lastly we deduce the $\mathbb{F}_p$ -Hopf algebra structure on $\mathrm{THH}_*(\mathrm{BP}\langle n \rangle/\mathrm{MU})$ by studying it as a $\mathbb{F}_p$ -Hopf algebra of $\mathrm{THH}_*(\mathbb{F}_p/\mathrm{MU})$.

Notation 1.5. *For the following Theorem A we explain the following pieces of notation. We let ku denote the connective complex K-theory spectrum. We define the infinite loop spaces*

$$U := \Omega^\infty \Sigma \mathrm{ku}, \quad SU := \Omega^\infty \Sigma^3 \mathrm{ku}, \quad \text{and} \quad BBSU := \Omega^\infty \Sigma^5 \mathrm{ku}$$

whose infinite loop space structures are determined by the (canonical) commutative ring structure on ku . For a prime p and positive integer n (except for $(p, n) = (2, 2)$) we define $\alpha(p, n)$ to be the unique power p^m of p such that the p -adic valuation of $v_p((1+p)^{p^m} - 1)$ of $(1+p)^{p^m} - 1$ is n . Lastly we define $U[\alpha(p, n)]$ to be the infinite loop space $SU \times_{\varphi(p, n)} S^1$ where encoded the $\alpha(p, n)$ -twist on S^1 . Note that $U[1] = 1$.

Theorem A. *The following statements are true.*

- *We have that $\mathbb{F}_2$ is an $\mathbb{E}_3$ -MU-Thom spectrum over U but is not an $\mathbb{E}_4$ - $\mathrm{MU}_2^\wedge$ -Thom spectrum over U .*
- *For all primes p and integers $n > 0$ other than (p, n) , we have that $\mathbb{Z}/p^n$ is an $\mathbb{E}_3$ - $\mathrm{MU}_{(p)}$ -Thom spectrum over $U[\alpha(p, n)]$ but is not an $\mathbb{E}_{2p+2}$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum over $U[\alpha(p, n)]$. Furthermore $\mathbb{F}_p$ is not an $\mathbb{E}_4$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum over U .*
- *We have that $\mathbb{Z}_{(p)}$ is an $\mathbb{E}_3$ - $\mathrm{MU}_{(p)}$ -Thom spectrum over $Su_p^\wedge$ but is not an $\mathbb{E}_{2p+2}$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum over $SU_p^\wedge$.*
- *We have that $\mathbb{Z}$ is an $\mathbb{E}_1$ -MU-Thom spectrum over SU but is not an $\mathbb{E}_2$ -MU-Thom spectrum over SU .*

- We have that ku is an $\mathbb{E}_1$ -MU-Thom spectrum over $BBSU$ but for all primes p that $\mathrm{ku}_p^\wedge$ is not an $\mathbb{E}_2$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum over $BBSU_p^\wedge$.

Proposition 1.6 (Proposition 4.8). *Fix $n > 1$ and R an $\mathbb{E}_{n+1}$ -ring. Let $f: X \rightarrow \mathrm{Pic}(R)$ denote a pointed $\mathbb{E}_n$ -space map. Let $T(f)$ denote its corresponding E_n - R -Thom spectrum and suppose that $T(f)$ admits an $\mathbb{E}_{n+1}$ - R -algebra structure not necessarily refining its $\mathbb{E}_n$ - R -algebra structure from being an $\mathbb{E}_n$ - R -Thom spectrum of f . Then there exists an equivalence*

$$\mathrm{THH}(T(f)/R) \simeq T(f)[BX]$$

of counital $\mathbb{E}_1$ -coalgebras in augmented $\mathbb{E}_{n-1}$ - $T(f)$ -algebras.

Conventions. We say category to mean ∞ -category $= (\infty, 1)$ -category. For X a space and R a spectrum we let $R[X]$ denote the homology spectrum $R \otimes \Sigma_+^\infty X$. We let Sp denote the category of spectra, let Spc denote the category of spaces ($= \mathrm{anima}$, ∞ -groupoids, weak homotopy types), and let Spc_* denote the category of pointed spaces. Spaces and spectra are assumed to neither be localized nor completed unless otherwise notated except for the Brown-Peterson spectra BP and their truncated variants $\mathrm{BP}\langle n \rangle$ which should be assumed to be p -localized. All spaces are assumed to be pointed and grouplike unless otherwise stated. All sheaves and stacks are assumed to be sheaves and stacks for the fpqc topology unless otherwise stated.

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2. PRELIMINARIES

In Section 2.1 we review generalized Thom spectra. In Subsection 2.2 we review coordinates of complex orientations of spectra as well as torsors for $\mathbb{E}_n$ -monoidal complex orientations. In Section 2.3 we review and calculate some Dyer-Lashof operations.

2.1. Generalized Thom Spectra. Our discussion largely follows [ABG⁺14a] on the ∞ -categorical approach to Thom spectra as well as [AB19] on universal properties of generalized Thom spectra.

For this subsection we fix R an $\mathbb{E}_1$ -ring. We begin by defining generalized Thom spectra, also known as R -Thom spectra. Let $\text{Pic}(R)$ denote the maximal sub- ∞ -groupoid of $\otimes_R$ -invertible objects in Mod_R .

Definition 2.1. *Fix a pointed space map $X \xrightarrow{f} \text{Pic}(R)$. The R -Thom spectrum $T(f)$ associated to the pointed space map f is the colimit*

$$\text{colim}(X \xrightarrow{f} \text{Pic}(R) \rightarrow \text{Mod}_R).$$

in Mod_R .

The process of associating a R -Thom spectrum to a pointed space map into $\text{Pic}(R)$ assembles into a colimit-preserving symmetric monoidal functor from the Cartesian symmetric monoidal overcategory $\text{Spc}_{*\text{Pic}(R)}$ of pointed spaces over the space $\text{Pic}(R)$ to Mod_R , as stated in [ABG18, Theorem 1.6]. A pointed space map $X \rightarrow \text{Pic}(R)$ where X is connected factors through the connected component $\text{BGL}_1(R)$ of $\text{Pic}(R)$ containing the identity R .

Multiplicative Coherence: The Thom spectrum functor, from $\mathbb{E}_n$ -monoidal category overcategory of pointed $\mathbb{E}_n$ -spaces over $\text{Pic}(R)$ to R -mod, is $\mathbb{E}_n$ -monoidal. [AB19, Corollary 3.2]

Universal Property: Let $f : X \rightarrow \text{Pic}(R)$ be an $\mathbb{E}_{n+1}$ -monoidal map with $\mathbb{E}_n$ - R -Thom spectrum $T(f)$. For A an $\mathbb{E}_n$ - R -algebra, there exists an equivalence of spaces between the space of $\mathbb{E}_n$ -monoidal maps from $T(f)$ to A and $\mathbb{E}_n$ -null-homotopies of

$$X \xrightarrow{f} \text{Pic}(R) \rightarrow \text{Pic}(A).$$

[AB19, Theorem 3.5] This was first proven in [ABG⁺14b] in the $\mathbb{E}_\infty$ -monoidal setting. See [HY20, discussion after Theorem 4.3].

Thom Isomorphism: ([AB19], Proposition 3.16) This was first proven for spectra in [Mah79, Theorem 1.2]) Let $f : X \rightarrow \text{Pic}(R)$ be an $\mathbb{E}_n$ -monoidal map. Then there is an equivalence

$$T(f) \otimes_R T(f) \simeq T(f) \otimes_R R[X] \cong T(f)[X]$$

of $\mathbb{E}_n$ - R -rings.

Base Change: For $A \rightarrow B$ an $\mathbb{E}_1$ -ring map and $X \xrightarrow{f} \text{BGL}_1(A)$ an $\mathbb{E}_n$ -space map then the $\mathbb{E}_n$ - B -Thom spectrum of the composition $X \rightarrow \text{BGL}_1(A) \rightarrow \text{BGL}_1(B)$ is $T(f) \otimes_A B$. If $X \rightarrow \text{BGL}_1(A) \rightarrow \text{BGL}_1(B)$ is null then there is an equivalence $T(f) \otimes_A B \simeq B[X]$ of $\mathbb{E}_n$ - B -algebras.

Intermediate Thom Spectrum Theorem: ([Dev24], Proposition 2.1.6) Let $X \xrightarrow{i} Y \rightarrow Z$ be a fiber sequence of $\mathbb{E}_n$ spaces with Z connected. Let $f : Y \rightarrow \text{Pic}(R)$ be an $\mathbb{E}_n$ -monoidal map. Then there exists an $\mathbb{E}_n$ -monoidal map $g : Z \rightarrow \text{BGL}_1(T(f \circ i))$ whose R -Thom spectrum $T(g)$ is equivalent to $T(f)$ as $\mathbb{E}_{n-1}$ -rings. The following diagram expresses the claims.

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \longrightarrow & Z \\ & \searrow f \circ i & \downarrow f & & \downarrow g \\ & & \text{BGL}_1(R) & \longrightarrow & \text{BGL}_1(T(f \circ i)) \end{array}$$

In this paper we often consider $\mathbb{E}_n$ -MU-Thom spectrum.

Example 2.2. Consider the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})}(\mathrm{MU}, \mathrm{MU})$ of $\mathbb{E}_n$ -ring maps $\mathrm{MU} \rightarrow \mathrm{MU}$. This mapping space is nonempty as it contains $\mathrm{id}_{\mathrm{MU}}$ which can be regarded as an $\mathbb{E}_n$ -ring map for all n . For all $n > 0$ there is an equivalence of mapping spaces

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})}(\mathrm{MU}, \mathrm{MU}) &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})}(\Sigma^\infty \mathrm{BU}, \mathrm{MU}) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Spc})}(\mathrm{BU}, \mathrm{GL}_1(\mathrm{MU})) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Spc})}(\mathrm{SU}, \mathrm{BGL}_1(\mathrm{MU})). \end{aligned}$$

where the first equivalence follows from the Thom isomorphism associated to the Thom spectrum MU , the second equivalence follows from definition of the spectrum of units $\mathrm{gl}_1(\mathrm{MU})$, the third equivalence follows from the (B, Σ^∞) adjunction, and the fourth and last equivalence follows from delooping once. This equivalence of mapping spaces is stated in [CM15, beginning of Section 6 through Proposition 6.2].

We learn that an $\mathbb{E}_n$ -ring map $\mathrm{MU} \rightarrow \mathrm{MU}$ determines an $\mathbb{E}_{n-1}$ - MU -Thom spectrum by considering the associated $\mathbb{E}_{n-1}$ -space map $\mathrm{SU} \rightarrow \mathrm{BGL}_1(\mathrm{MU})$ and taking its associated $\mathbb{E}_{n-1}$ - MU -Thom spectrum.

2.2. Complex Orientations. In this subsection we review complex orientations of $\mathbb{E}_n$ -ring spectra and their corresponding coordinates. We also review $\mathbb{E}_1$ - R -Thom spectra of maps $\mathrm{SU} \rightarrow \mathrm{BGL}_1(R)$ for R an even $\mathbb{E}_\infty$ -ring.

Let R be a homotopy commutative even ring spectrum. Recall that a complex orientation of R is a choice of coordinates of the formal group $\mathrm{Spf}(R^*(\mathbb{CP}^\infty))$ compatible with $\mathbb{G}_m$ -action. Concretely, the coordinates of a complex orientation of R is the same data as a power series

$$x + \sum_i a_i x^{i+1}$$

for $a_i \in \pi_{2i}(R)$ and x denoting the standard complex orientation of MU . The coordinates of R form a $\mathbb{G}_{\mathrm{inv}}(R)$ -torsor, where $\mathbb{G}_{\mathrm{inv}}(R)$ denotes the automorphism group of the unit disc, with action $\mathbb{G}_{\mathrm{inv}}(R)$ acting on coordinates by composition.

We recall that an equivalent definition of a complex orientation of R is a homotopy commutative ring map $\mathrm{MU} \rightarrow R$. By the classical Thom isomorphism we have an equivalence

$$\mathrm{Map}_{\mathrm{hoRing}}(\mathrm{MU}, R) \cong \mathrm{Map}_{\mathrm{hoRing}}(\Sigma_+^\infty \mathrm{BU}, R)$$

of mapping sets where hoRing denotes the 1-category of homotopy commutative ring spectra. The infinite loop space BU induces a group structure on $\mathrm{Map}_{\mathrm{hoRing}}(\Sigma_+^\infty \mathrm{BU}, R)$ that makes $\mathrm{Map}_{\mathrm{hoRing}}(\mathrm{MU}, R)$ into an $\mathrm{Map}_{\mathrm{hoRing}}(\Sigma_+^\infty \mathrm{BU}, R)$ -torsor.

We remark that the universal property of GL_1 gives a correspondence of mapping sets between $\mathrm{Map}_{\mathrm{hoRing}}(\Sigma_+^\infty \mathrm{BU}, R)$ and the set of maps of spaces up to homotopy from BU to $\mathrm{GL}_1(R)$. The following Proposition 2.3 compares the coordinates identifying complex orientation and their corresponding maps of spaces up to homotopy from BU to $\mathrm{GL}_1(R)$.

Proposition 2.3 ([BSS20], Section 5.1). *Let R be an even homotopy commutative ring spectrum. A homotopy ring map $\mathrm{MU} \rightarrow R$ with coordinates $a_i \in \pi_{2i}(R)$ determines a map of spaces $\mathrm{BU} \rightarrow \mathrm{GL}_1(R)$ that sends $x_i \in \pi_*(\mathrm{BU}) \cong \mathbb{Z}1$ with $\deg(x_i) = 2i$ to $a_i \in \pi_{2i}(R)$ in $\mathrm{GL}_1(R)$.*

We recall facts on multiplicative coherence of complex orientations of even $\mathbb{E}_1$ -rings. In [CM15], it is determined that complex orientations of even ring spectra always admit $\mathbb{E}_2$ -monoidal refinements, yet generally they may or may not admit $\mathbb{E}_4$ -monoidal refinements.

In [CM15, Section 6], $\mathbb{E}_n$ -monoidal complex orientations of MU are studied by looking at the forgetful maps

$$\pi_0 \operatorname{Map}_{\mathbb{E}_n\text{-Rings}}(\Sigma_+^\infty \operatorname{BU}, \operatorname{MU}) \rightarrow \operatorname{Map}_{\operatorname{hoRing}}(\Sigma_+^\infty \operatorname{BU}, \operatorname{MU})$$

of terms of maps on cohomology

$$H^{*+n}(B^n \operatorname{BU}; \mathbb{Z}) \rightarrow H^*(\operatorname{BU}(1); \mathbb{Z})$$

where $B^n U$ denotes the n -fold classifying space of U . In particular, for $n = 4$, they study maps $H^{*+4}(\operatorname{BU}\langle 6 \rangle; \mathbb{Z}) \rightarrow H^*(\operatorname{BU}(1); \mathbb{Z})$. For the following we notate $H^*(\operatorname{BU}\langle 6 \rangle; \mathbb{Z}) \cong \mathbb{Z}[y_3, y_4, y_5, \dots]$ with $\deg(y_i) = 2i$ and $H^*(\operatorname{BU}(1); \mathbb{Z}) \cong \mathbb{Z}[x]$ with $\deg(x) = 2$. Their results are stated in Proposition 2.4.

Proposition 2.4 ([CM15], Proposition 6.4). *The maps $H^{*+4}(\operatorname{BU}\langle 6 \rangle; \mathbb{Z}) \rightarrow H^*(\operatorname{BU}(1); \mathbb{Z})$ are as follows. The generator $y_{m+2} \in H^{2m+4}(\operatorname{BU}\langle 6 \rangle; \mathbb{Z})$ is sent to the following:*

$$\begin{cases} (-1)^{p^t} p^{t-1} x^{p^t-1} & \text{if } m+1 = p^t \text{ for some prime } p, t > 0 \\ (-1)^{m+1} (m+1) x^m & \text{otherwise} \end{cases}$$

We explain how to use Proposition 2.4 to study coordinates of $\mathbb{E}_4$ -ring automorphisms of MU. Given one $\mathbb{E}_4$ -ring map with coordinates $x + a_1 x^2 + a_2 x^3 + \dots$, then we can add any $b = p^{t-1}$ or $b = m+1$ (depending on if i is one less than a power of a prime). The result is an $\mathbb{E}_4$ -ring map $\operatorname{MU} \rightarrow \operatorname{MU}$ with coordinates

$$x + a_1 x^2 + a_2 x^3 + \dots + a_{m-2} x^{m-1} + (a_{m-1} + b) x^m + \dots$$

where the higher coordinates aren't described by the torsor structure. This is because at the $2i$ th Postnikov stage $\tau_{\leq 2*} B^4 GL_1(\operatorname{MU})$, the choices of lifts to the next stage $\tau_{\leq 2*+2} B^4 GL_1(\operatorname{MU})$ for a torsor for $H^{2*+2}(\operatorname{BU}\langle 6 \rangle), \pi_{2*}(\operatorname{MU})$ with torsor action described by Proposition 2.4.

We recall results from [BSS20] on how $\mathbb{E}_2$ -monoidal complex orientations $\operatorname{MU} \rightarrow R$ for R even give rise to $\mathbb{E}_1$ - R -Thom spectra. In particular we recall the following Proposition 2.5 which we've stated in a mildly generalized form.

Proposition 2.5 ([BSS20], Theorem 1.5). *For $u_1, \dots$, a sequence of even-indexed classes in $\pi_*(R)$, there exists an $\mathbb{E}_1$ -monoidal map of spaces $\operatorname{SU} \rightarrow BGL_1(R)$ with $\mathbb{E}_1$ - R -Thom spectrum isomorphic to an $\mathbb{E}_1$ - R -algebra $R/(u_1, \dots)$.*

As a consequence of the Thom isomorphism of $\mathbb{E}_n$ -algebras and Proposition 2.5, we have the following Proposition 2.6.

Proposition 2.6. *For $(a_1, a_2, \dots)$ the coordinates of an $\mathbb{E}_n$ -ring map $\operatorname{MU} \rightarrow \operatorname{MU}$, the corresponding $\mathbb{E}_{n-1}$ -space map $\overline{f} : \operatorname{SU} \rightarrow BGL_1(\operatorname{MU})$ has $\mathbb{E}_{n-1}$ - MU -Thom spectrum isomorphic to $\operatorname{MU}/(a_1, a_2, \dots)$.*

2.3. Dyer-Lashof operations. In this subsection we review Dyer-Lashof operations and perform some computations of Dyer-Lashof operations. Let n be a fixed positive integer.

In Lemma 2.8 we will study Dyer-Lashof operations on some $\mathbb{F}_p$ -algebras. Recall that $\mathbb{F}_p$, $\mathbb{Z}$, and ku each have canonical $\mathbb{E}_\infty$ -MU-algebra structures. These allow us to consider canonical $\mathbb{E}_n$ -MU-algebra structures on $R \otimes_{\operatorname{MU}} S$ for $R, S \in \{\mathbb{F}_p, \mathbb{Z}, \operatorname{ku}\}$ by [HW22, Section 2.4] and using the forgetful functor from $\mathbb{E}_\infty$ -algebras to $\mathbb{E}_n$ -algebras.

Proposition 2.7 ([HW22], Lemma 2.4.1). *The $\mathbb{E}_1$ -MU-enveloping algebra $\mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{F}_p$ of $\mathbb{F}_p$, with its canonical $\mathbb{E}_2$ - $\mathbb{F}_p$ -algebra structure, has homotopy ring*

$$\Lambda_{\mathbb{F}_p*}[\sigma x_0, \sigma x_1, \dots]$$

with $|\sigma x_i| = 2i + 1$ and Dyer-Lashof operations

$$Q^{2^{t+1}}(\sigma x_{2^t}) = \sigma x_{2^{t+1}} \text{ mod units} \quad \text{for all } t \geq 0, p = 2$$

$$Q^{t^i}(\sigma x_{2^{p^t-2}}) = \sigma x_{2^{p^{t+1}-2}} \text{ mod units} \quad \text{for all } t \geq 0, p > 2$$

$$Q^{2^{j+2}}(\sigma x_j) = \sigma x_{2^{j+1}} \text{ mod units decomposables} \quad \text{for } p = 2$$

$$Q^{j+1}(\sigma x_j) = \sigma x_{pj+2p-2} \text{ mod units decomposables} \quad \text{for } p > 2$$

Lemma 2.8. *Let p be any prime. The sequence of $\mathbb{E}_\infty$ -MU- $\mathbb{Z}$ -algebras*

$$\mathbb{Z} \otimes_{\mathrm{MU}} \mathrm{ku} \rightarrow \mathbb{Z} \otimes_{\mathrm{MU}} \mathbb{Z} \rightarrow \mathbb{Z} \otimes_{\mathrm{MU}} \mathbb{F}_p,$$

after applying π_* becomes

$$\Lambda_{\mathrm{ku}*}[\sigma x_1, \sigma x_2, \dots] \rightarrow \Lambda_{\mathbb{Z}*}[\sigma x_1, \sigma x_2, \dots] \rightarrow \Lambda_{\mathbb{F}_p*}[\sigma x_1, \sigma x_2, \dots]$$

with $|\sigma x_i| = 2i + 1$. Likewise the sequence of $\mathbb{E}_\infty$ -MU- $\mathbb{F}_p$ -algebras

$$\mathbb{F}_p \otimes_{\mathrm{MU}} \mathrm{ku} \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{Z} \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{F}_p,$$

after applying π_* becomes

$$\Lambda_{\mathrm{ku}*}[\sigma x_0, \sigma x_1, \dots] \rightarrow \Lambda_{\mathbb{Z}*}[\sigma x_0, \sigma x_1, \dots] \rightarrow \Lambda_{\mathbb{F}_p*}[\sigma x_0, \sigma x_1, \dots]$$

with $|\sigma x_i| = 2i + 1$. Their $\mathbb{E}_2$ Dyer-Lashof operations are determined mod decomposables (and mod units away from the σv_i 's) from the previous proposition.

Proof. We justify the homotopy ring calculations. By [Ang08, Proposition 3.6], and considering how ku is a quotient of MU by regular ideal $(x_2, x_3, \dots)$, we have that

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{MU}} \mathrm{ku}) \cong \Lambda_{\mathrm{ku}*}[\sigma x_2, \sigma x_3, \dots]$$

with $|\sigma x_i| = 2i + 1$. This strategy was used to compute $\pi_*(\mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{F}_p)$ in the proof of [HW22, Lemma 2.4.1]. To obtain the other homotopy ring calculations, we recall how $\mathbb{Z}$ is obtained as a quotient of ku by the regular sequence (x_2) and how $\mathbb{F}_p$ is obtained as a quotient of $\mathbb{Z}$ by the regular sequence (p) .

In all cases, we are considering cofiber sequences associated to a regular quotient $R \rightarrow S$ by $x \in R$. We also have some A , a regular quotient of MU by the ideal $(a_1, a_2, \dots)$ of the form

$$A \otimes_{\mathrm{MU}} R \xrightarrow{1 \otimes x} A \otimes_{\mathrm{MU}} R \rightarrow A \otimes_{\mathrm{MU}} S,$$

which allows us to conclude

$$\begin{aligned} \pi_*(A \otimes_{\mathrm{MU}} S) &= \pi_*(\mathrm{cofib}(A \otimes_{\mathrm{MU}} R \xrightarrow{1 \otimes x} A \otimes_{\mathrm{MU}} R)) \\ &= \pi_*(\mathrm{cofib}(\Lambda_{R*}[\sigma a_1, \sigma a_2, \dots] \xrightarrow{x} \Lambda_{R*}[\sigma a_1, \sigma a_2, \dots])) \\ &= \Lambda_{S*}[\sigma a_1, \sigma a_2, \dots]. \end{aligned}$$

We prove the listed Dyer-Lashof operation identities. Consider the sequence of $\mathbb{E}_\infty$ - $\mathbb{F}_p$ -algebras

$$\mathbb{F}_p \otimes_{\mathrm{MU}} \mathrm{ku} \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{Z} \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{F}_p.$$

The result follows from the listed Dyer-Lashof operation identities of Proposition 2.7 and naturality of Dyer-Lashof operations. $\square$

Now that we've studied the Q_1 Dyer-Lashof operations we will need for certain $\mathbb{E}_2$ -MU- $\mathbb{F}_p$ -algebras, we move on to studying the Q_1 Dyer-Lashof operations for certain $\mathbb{F}_p$ -homologies of spaces.

Proposition 2.9 ([Koc71], Theorem 6). *In $\pi_{ast}(\mathbb{F}_p[\text{BU}]) \cong \mathbb{F}_p[b_1, b_2, \dots]$ we have the following identities for the Dyer-Lashof operations:*

$$\begin{aligned} Q^{2j}(b_n) &= \binom{j-1}{n} b_{n+j} \text{ mod units} && \text{for all } n \geq 1, p = 2 \\ Q^j(b_n) &= \binom{j-1}{n} b_{n+(p-1)j} \text{ mod units} && \text{for all } n \geq 1, p > 2 \end{aligned}$$

Proposition 2.10 ([Koc71]). *In $\pi_*(\mathbb{F}_p[U]) \cong \Lambda_{\mathbb{F}_p}(\sigma b_0, \sigma b_1, \dots)$ and $\pi_*(\mathbb{F}_p[SU]) \cong \Lambda_{\mathbb{F}_p}(\sigma b_1, \sigma b_2, \dots)$ with $|\sigma b_i| = 2i + i$ we have the following identities for the Dyer-Lashof operations mod decomposables:*

$$\begin{aligned} Q^{2j}(\sigma b_n) &= \binom{j-1}{n} \sigma b_{n+j} \text{ mod units} && \text{for all } n \geq 1, p = 2 \\ Q^j(\sigma b_n) &= \binom{j-1}{n} \sigma b_{n+(p-1)j} \text{ mod units} && \text{for all } n \geq 1, p > 2 \end{aligned}$$

Proposition 2.11. *We have the following identities for the $\mathbb{E}_2$ Dyer-Lashof on the polynomial classes b_i 's operations mod decomposables:*

$$\begin{aligned} Q^{2n+2}(\sigma b_n) &= \sigma b_{2n+1} \text{ mod units} && \text{for all } n \geq 1, p = 2 \\ Q^{n+1}(\sigma b_n) &= \sigma b_{pn+(p-1)} \text{ mod units} && \text{for all } n \geq 1, p > 2. \end{aligned}$$

We have the following identities for the $\mathbb{E}_4$ Dyer-Lashof on the exterior classes σb_i 's operations mod decomposables:

$$\begin{aligned} Q^4(\sigma b_0) &= \sigma b_2 \text{ mod units} && \text{for } p = 2 \\ Q^2(\sigma b_0) &= \sigma b_{2p-2} \text{ mod units} && \text{for all } p > 2. \end{aligned}$$

We have the following identities for the $\mathbb{E}_{2p+2}$ ($\mathbb{E}_4$ for $p = 2$) Dyer-Lashof operations on the exterior classes σb_i 's mod decomposables:

$$\begin{aligned} Q^8(\sigma b_0) &= \sigma b_5 \text{ mod units} && \text{for } p = 2 \\ Q^{2p}(\sigma b_{p-1}) &= \sigma b_{2p^2-p-1} \text{ mod units} && \text{for all } p > 2. \end{aligned}$$

3. GENERALIZED THOM SPECTRA AND THEOREM A

Notation 3.1. *We make the following definitions of $\mathbb{E}_\infty$ -spaces.*

- We define

$$U := \Omega^\infty \Sigma \text{ku}, \quad SU := \Omega^\infty \Sigma^3 \text{ku}, \quad \text{and} \quad BBSU := \Omega^\infty \Sigma^5 \text{ku}$$

- We define the $\mathbb{E}_\infty$ -ring space $U[n]$ via the pullback square

$$\begin{array}{ccc} U[n] & \longrightarrow & U \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{[n]} & S^1 \end{array}$$

- For every prime p and integer $n > 0$, other than $(p, n) = (2, 2)$, we define $\alpha(p, n)$ to be the unique power p^m of p such that the p -adic valuation of $v_p((1+p)^{p^m} - 1)$ of $(1+p)^{p^m} - 1$ is n . Concretely

$$\alpha(p, n) := \begin{cases} p^{n-1} & \text{if } p > 2 \\ 1 & \text{if } p = 2 \text{ and } n = 1, \\ 2^{n-2} & \text{if } p = 2 \text{ and } n > 2 \end{cases}$$

and we will consider the $\mathbb{E}_\infty$ -spaces $U[\alpha(p, n)]$.

Theorem 3.2 (Theorem A). *We have the following existence statements:*

- (i) *There exists an $\mathbb{E}_3$ -space map*

$$U \rightarrow BGL_1(\mathrm{MU})$$

whose associated $\mathbb{E}_3$ -MU-Thom spectrum is isomorphic to $\mathbb{F}_2$.

- (ii) *For every prime p and positive integer n (except for $(p, n) = (2, 2)$) there exists a p -local $\mathbb{E}_3$ -space map*

$$U[\alpha(p, n)] \rightarrow BGL_1(\mathrm{MU}_{(p)}),$$

whose associated $\mathbb{E}_3$ - $\mathrm{MU}_{(p)}$ -Thom spectrum is isomorphic to $\mathbb{Z}/p^n$.

- (iii) *For every prime p there exists a p -local $\mathbb{E}_3$ -space map*

$$SU \rightarrow BGL_1(\mathrm{MU}_{(p)})$$

whose associated $\mathbb{E}_3$ - $\mathrm{MU}_{(p)}$ -Thom spectrum is isomorphic to $\mathbb{Z}_{(p)}$.

- (iv) *There exists an $\mathbb{E}_1$ -space map*

$$BBSU \rightarrow BGL_1(\mathrm{MU})$$

whose associated $\mathbb{E}_1$ -MU-Thom spectrum is isomorphic to ku .

We also have the following non-existence statements:

- (v) *For all primes p , there does not exist a p -complete $\mathbb{E}_4$ -space map*

$$U[\alpha(p, 1)] \rightarrow BGL_1(\mathrm{MU}_p^\wedge)$$

whose associated $\mathbb{E}_4$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{F}_p$.

- (vi) *For all primes p and integers $n > 1$ other than $(p, n) = (2, 2)$, there does not exist a p -complete $\mathbb{E}_{2p+2}$ -space map*

$$U[\alpha(p, n)] \rightarrow BGL_1(\mathrm{MU}_p^\wedge)$$

whose associated $\mathbb{E}_{2p+2}$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{Z}/p^n$.

- (vii) *For all primes p there does not exist a p -complete $\mathbb{E}_4$ -space map*

$$SU \rightarrow BGL_1(\mathrm{MU}_p^\wedge)$$

whose associated $\mathbb{E}_{2p+2}$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{Z}_p^\wedge$.

(viii) There does not exist a $\mathbb{E}_2$ -space map

$$SU \rightarrow BGL_1(\mathrm{MU})$$

whose associated $\mathbb{E}_2$ -MU-Thom spectrum is isomorphic to $\mathbb{Z}$.

(ix) For all primes p there does not exist a p -complete $\mathbb{E}_2$ -space map

$$BBSU \rightarrow BGL_1(\mathrm{MU}_p^\wedge)$$

whose associated $\mathbb{E}_2$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathrm{ku}_p^\wedge$.

3.1. Existence Statements for Theorem A.

Lemma 3.3. *The $\mathbb{E}_3$ -MU-Thom spectrum $T(f)$ of an $\mathbb{E}_3$ -space map $f : SU \rightarrow BGL_1(\mathrm{MU})$ orients $\mathbb{Z}$ via its 0th Postnikov truncation map. In other words, the 0th Postnikov truncation map is a map*

$$T(f) \xrightarrow{\pi_0} \pi_0(T(f)) \cong \mathbb{Z}$$

of $\mathbb{E}_3$ -MU-Thom spectra.

Proof. Because SU is 2-connected, $\pi_0(T(f)) \cong \pi_0(\mathrm{MU}) \cong \mathbb{Z}$ as MU-modules. The fact that the 0th Postnikov truncation map is $\mathbb{E}_3$ -monoidal follows from $\pi_0(-)$ being a symmetric monoidal functor. $\square$

Lemma 3.4. *For every prime p there exists a $\mathbb{E}_3$ -space map $f_p : SU \rightarrow BGL_1(\mathrm{MU})$ with coordinates $(a_1, a_2, \dots)$ satisfying the following two properties:*

- (1) *We have $a_{p-1} = x_{p-1}$.*
- (2) *For i such that p doesn't divide $i + 1$, the image of a_i mod p and decomposables is nonzero.*

Proof. We construct such an f_p one coordinate at a time. By Proposition 2.4, the indices i for which

$$H^{i+4}(BU\langle 6 \rangle; \mathbb{F}_p) \rightarrow H^i(BU(1); \mathbb{F}_p)$$

is surjective includes the indices $i + 1 = p$ and i such that p doesn't divide $i + 1$. Therefore the coordinates a_i for these indices i may be freely chosen and in particular may be chosen to satisfy the two properties. $\square$

For the remainder of the section, we fix a prime p . Furthermore, we fix $\mathbb{E}_4$ -ring map $f : \mathrm{MU} \rightarrow \mathrm{MU}$ as describe in Lemma 3.4. We fix its p -localize it so that is an E_4 -ring map $\mathrm{MU}_{(p)} \rightarrow \mathrm{MU}_{(p)}$ and corresponds to an $\mathbb{E}_3$ -space map $SU_{(p)} \rightarrow BGL_1(\mathrm{MU}_{(p)})$.

Lemma 3.5. *The coordinates a_i of f_p satisfy $a_i \equiv u_i x_i$ mod decomposables for $u_i \in (\mathbb{Z}_{(p)})^\times$.*

Proof. We are tasked to show that for i such that p divides $i + 1$ that the desired claim is true of a_i . We will induct on the collection of such i from smallest upwards. The base case of the induction is the case of $i + 1 = p$ where we have $a_{p-1} = x_{p-1}$.

Suppose for the sake of induction that for all $j < i$ that a_j satisfied the desired property and p divides $j + 1$ so that we seek to show that a_j satisfies the desired property. The image of a_j is also mapped onto by the $\mathbb{E}_1$ -power operation Q^{2j+2} (at the prime 2) or Q^{j+1} (at odd primes) by some class a_k . By inductive hypothesis, a_k satisfies the desired claim, thus a_j satisfies the desired claim by naturality of the $\mathbb{E}_1$ -power operation. $\square$

Let $T(p)$ denote the $\mathbb{E}_3 - \mathrm{MU}_{(p)}$ -THom spectrum associated to $T(f_p)$.

Lemma 3.6. *The induced $\mathbb{F}_p$ -homology map on the 0th-Postnikov truncation map*

$$\mathbb{F}_p \otimes_{\mathrm{MU}} \pi_0(-) : \mathbb{F}_p \otimes T(f) \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{Z}$$

is an equivalence of $\mathbb{E}_3\text{-}\mathbb{F}_p\text{-}\mathrm{MU}_{(p)}$ -algebras.

Proof. By the Thom isomorphism, there exists an equivalence

$$\mathbb{F}_p[\mathrm{SU}] \rightarrow \mathbb{F}_p \otimes T(f_p)$$

of $\mathbb{E}_3\text{-}\mathbb{F}_p\text{-}\mathrm{MU}_{(p)}$ -algebras, therefore we will show that the composition

$$\mathbb{F}_p[\mathrm{SU}] \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{Z}$$

is an equivalence $\mathbb{E}_3\text{-}\mathbb{F}_p\text{-}\mathrm{MU}_{(p)}$ -algebras. By the Whitehead Theorem, it is sufficient to show that the associated map of homotopy rings

$$\Lambda_{\mathbb{F}_p}[\sigma c_1, \sigma c_2, \dots] \rightarrow \Lambda_{\mathbb{F}_p}[\sigma x_1, \sigma x_2]$$

is an isomorphism. This follows from Lemma 3.5. $\square$

A similar proof to that of the above proof of Lemma 3.6 shows that the induces rational homology map on the 0-th Postnikov truncation map is an equivalence of $\mathbb{E}_3\text{-}\mathbb{Q}\text{-}\mathrm{MU}_{(p)}$ -algebras.

We now prove Part (iii) of Theorem 3.2 whose statement we recall now. There exists $\mathbb{E}_3$ -space map $f_p : \mathrm{SU}_{(p)} \rightarrow \mathrm{BGL}_1(\mathrm{MU})$ whose $\mathbb{E}_3\text{-}\mathrm{MU}$ -Thom spectra $T(f)$ has the property that the Postnikov truncation map $T(f) \rightarrow \mathbb{Z}_{(p)}$ is an equivalence of $\mathbb{E}_3\text{-}\mathrm{MU}_{(p)}$ -algebras.

Proof of Theorem 3.2 Part (iii). We will show that the 0th-Postnikov truncation map $T(f) \rightarrow \mathbb{Z}_{(p)}$ is an equivalence of $\mathbb{E}_3\text{-}\mathrm{MU}_{(p)}$ -algebras. By connectivity and arithmetic fracture square considerations, it is sufficient to show that the 0th-Postnikov truncation map induces isomorphisms of $\mathbb{E}_3\text{-}\mathbb{F}_p\text{-}\mathrm{MU}_{(p)}$ -algebras upon taking $\mathbb{F}_p$ homology as well as isomorphisms of $\mathbb{E}_3\text{-}\mathbb{Q}\text{-}\mathrm{MU}_{(p)}$ -algebras upon taking $\mathbb{Q}$ homology. This was studied in Lemma 3.6 for $\mathbb{F}_p$ homology and in subsequent discussion for $\mathbb{Q}$ homology. $\square$

Consider the standard $\mathbb{E}_\infty$ -orientation $\mathrm{MU} \rightarrow \mathrm{ku}$ so that there is an induced map $\theta : \mathrm{BGL}_1(\mathrm{MU}) \rightarrow \mathrm{BGL}_1(\mathrm{ku})$.

Lemma 3.7. *A space map $\mathrm{BU}\langle 6 \rangle \rightarrow \mathrm{fib}(B\theta)$ forming the dotted lift in the diagram*

$$\begin{array}{ccccc} & & \mathrm{fib}(B\theta) & & \\ & \nearrow & \downarrow & & \\ \mathrm{BU}\langle 6 \rangle & \longrightarrow & B^2\mathrm{GL}_1(\mathrm{MU}) & \xrightarrow{B\theta} & B^2\mathrm{GL}_1(\mathrm{ku}) \end{array}$$

of spaces exists. Furthermore, the choice of space map $\mathrm{BU}\langle 6 \rangle \rightarrow \mathrm{fib}(B\theta)$ determines an $\mathbb{E}_1$ -space map $f : \mathrm{BBSU} \rightarrow \mathrm{BGL}_1(\mathrm{MU})$ that factors (in $\mathbb{E}_1$ -spaces) through $\mathrm{fib}(\theta)$ and thus determines an $\mathbb{E}_1\text{-}\mathrm{MU}$ -algebra map $T(f) \rightarrow \mathrm{ku}$.

Proof. By obstruction theory there exists a space map $\mathrm{BU}\langle 6 \rangle \rightarrow B^2\mathrm{GL}_1(\mathrm{MU})$ by degree/evenness reasons. We argue that as one builds a map that one can also perform the dotted extension at each Postnikov stage. Observe that $\pi_*(\mathrm{fib}(B\theta))$ is even from analysis of its associated long exact sequence in homotopy rings, and $\pi_*(B^2\mathrm{GL}_1(\mathrm{MU})) \cong \pi_{*-2}(\mathrm{GL}_1(\mathrm{MU}))$, so $\pi_*(B^2\mathrm{GL}_1(\mathrm{MU}))$ is even. Thus associated obstruction groups

$$H^{2*+1}(\mathrm{BU}\langle 6 \rangle; \pi_*(\mathrm{fib}(B\theta)))$$

to the extension problem of vanish for degree reasons. The resulting claim about orienting ku follows from the universal property of Thom spectra and that the composition $\mathrm{BU}\langle 6 \rangle \rightarrow B^2 \mathrm{GL}_1(\mathrm{ku})$ factors through $\mathrm{fib}(B\theta)$ and therefore determines the desired $\mathbb{E}_1$ -nullhomotopy. $\square$

Lemma 3.8. *There exists $\mathbb{E}_1$ -space map $f : BBSU \rightarrow BGL_1(\mathrm{MU})$ whose $\mathbb{E}_1$ -MU-Thom spectrum $T(f)$ orients ku (say via $f : T(f) \rightarrow \mathrm{ku}$) in such a way so that the induced map (after the canonical base change $\mathrm{ku} \rightarrow \mathbb{Z}$) on $\mathbb{Z}$ -homology*

$$\mathbb{Z} \otimes_{\mathrm{MU}} f : \mathbb{Z}[T(f)] \rightarrow \mathbb{Z} \otimes_{\mathrm{MU}} \mathrm{ku}$$

has the property that each class σx_n in the homotopy ring

$$\pi_*(\mathbb{Z} \otimes_{\mathrm{MU}} \mathrm{ku}) \cong \Lambda_{\mathbb{Z}}(\sigma x_2, \sigma x_3, \dots)$$

with $|\sigma x_n| = 2n + 1$ is surjected upon by the map $\pi_*(\mathbb{Z} \otimes_{\mathrm{MU}} f)$.

Proof. It suffices to show that the images may be chosen freely, in other words, that the torsor of lifts that occurred in the lifting in the proof of Lemma 3.8 has transitive action. We seek to understand $\mathrm{fib}(B\theta)^2(\mathrm{BU}\langle 6 \rangle)$. The Atiyah-Hirzebruch spectral sequence

$$E_2^{s,t} = H^s(\mathrm{BU}\langle 6 \rangle; \pi_t(\mathrm{fib} B\theta)) \Rightarrow (fib(\theta))^{s-t}(\mathrm{BU}\langle 6 \rangle)$$

has $E_2 = E_\infty$ -pages by evenness. This gives a non-canonical isomorphism

$$(fib(\theta))^q(\mathrm{BU}\langle 6 \rangle) \cong \bigoplus_{m > m} H^{m+q}(\mathrm{BU}\langle 6 \rangle) \otimes \pi_m(\mathrm{fib}(B\theta)).$$

and it remains to show that the composition

$$H^{2*+2}(\mathrm{BU}\langle 6 \rangle; \pi_{2*} \mathrm{fib}(\theta)) \rightarrow H^{2*+1}(BBSU; \pi_{2*} \mathrm{fib}(\theta)) \rightarrow H^{2*+1}(BBSU; \pi_{2*+1}(Z \otimes_{\mathrm{MU}} \mathrm{ku}))$$

is surjective. This follows from the first map being induced by the suspension map $H^{*+1}(\mathrm{BU}\langle 6 \rangle) \rightarrow H^*(BBSU)$ and the second map being induced from the map $\Sigma \mathrm{fib}(\theta) \rightarrow \mathrm{GL}_1(Z \otimes_{\mathrm{MU}} \mathrm{ku})$. $\square$

We are now prepared to prove Theorem 3.2 part (iv) whose statement we recall now. There exists an $\mathbb{E}_1$ -space map $BBSU \rightarrow BGL_1(\mathrm{MU})$ whose associated $\mathbb{E}_1$ -MU-Thom spectrum is isomorphic to ku .

proof of Theorem 3.2 Part (iv). We will any choice of $\mathbb{E}_1$ -space map $f : BBSU \rightarrow BGL_1(\mathrm{ku})$ from Lemma 3.8 has the property that the corresponding orientation $T(f) \rightarrow \mathrm{ku}$ is an $\mathbb{E}_1$ -MU-algebra isomorphism

By connectivity and Whitehead Theorem considerations, it suffices to show that the induced map

$$\mathbb{Z}[T(f)] \rightarrow \mathbb{Z} \otimes_{\mathrm{MU}} \mathrm{ku}$$

is an isomorphism on homotopy rings. By the Thom isomorphism says

$$\mathbb{Z}[T(f)] \cong \mathbb{Z}[BBSU]$$

as $\mathbb{E}_1$ -MU-algebras, and $\mathbb{Z}[BBSU]$ has its homotopy ring abstractly isomorphic to that of $\mathbb{Z} \otimes_{\mathrm{MU}} \mathrm{ku}$. To see the $\mathbb{Z}$ -homology map is indeed an isomorphism, it suffices to know it is surjective on homotopy rings which was concluded in Lemma 3.8. $\square$

We move on to proving the $\mathbb{E}_3$ -MU-Thom spectrum presentation of $\mathbb{F}_2$ as stated in Theorem 3.2 part (i) as well as the $\mathbb{E}_3$ - $MU_{(p)}$ -Thom spectrum presentations of $\mathbb{Z}/p^n$ (but not $\mathbb{Z}/4$) as stated in Theorem 3.2 part (ii).

For the following lemma we note the following fact:

$$\pi_1(BGL_1(MU_{(p)})) \cong \pi_0(GL_1(MU_{(p)})) \cong \mathbb{Z}_{(p)}^\times.$$

Lemma 3.9. *There exists $\mathbb{E}_3$ -space map $f : SU \rightarrow BGL_1(MU_{(p)})$ such that the p -localized space map f_p has the property that $T(f_p) \simeq \mathbb{Z}_{(p)}$ as $\mathbb{E}_3 - MU_{(p)}$ -algebras. Furthermore, f satisfies the property that it factorized through U in such a way that the inclusion $S^1 \hookrightarrow U$ gives the map*

$$U(1) \hookrightarrow U \rightarrow SU_{(p)} \xrightarrow{f_p} BGL_1(MU_{(p)})$$

that corresponds to $1 + p \in \pi_1(BGL_1(MU_{(p)})) \cong \mathbb{Z}_{(p)}^\times$.

Proof. To produce the desired factorization is equivalent to producing the dotted map in the diagram

$$\begin{array}{ccc} BSU & & \\ \uparrow & \searrow \cdots & \\ BU\langle 6 \rangle_{(p)} & \xrightarrow{B^3 f_p} & B^4 GL_1(MU_{(p)}) \end{array}$$

of spaces. The associated obstruction groups

$$H^{2*+1}(BSU_{(p)}, BU\langle 6 \rangle_{(p)}; \pi_{2*}(B^4 GL_1(MU_{(p)})))$$

vanish for degree reasons, so the factorization exists. The ability to choose the class

$$S_{(p)}^1 = U(1)_{(p)} \hookrightarrow U_{(p)} \rightarrow BGL_1(MU_{(p)})$$

up to homotopy comes from the choice permitted in constructing the first Postnikov stage

$$[BSU, K(\mathbb{Z}_{(p)}^4, 4)] \cong H^4(BSU; \mathbb{Z}_{(p)}^\times) \cong \mathbb{Z}_{(p)}^\times.$$

Finally, one checks that choosing $1 + p$ for the first Postnikov stage produces a map $SU_{(p)} \rightarrow BGL_1(MU_{(p)})$ satisfying the criteria of Lemma 3.4. $\square$

Corollary 3.10. *The composition $\mathbb{E}_3$ -space map composition*

$$f_{p,n} := U[\alpha(p, n)] \rightarrow U \rightarrow BGL_1(MU_{(p)}),$$

with $U \rightarrow BGL_1(MU_{(p)})$ constructed from Lemma 3.9, has the property that

$$U(1)_{(p)} \hookrightarrow U[\alpha(p, n)] \rightarrow BGL_1(MU_{(p)})$$

corresponds to $(1 + p)^\alpha \in \pi_1(BGL_1(MU_{(p)})) \cong \mathbb{Z}_{(p)}^\times$.

We prove Theorem 3.2 part (ii) whose statement we recall. For every prime p and positive integer n (except for $(p, n) = (2, 2)$) the p -local $\mathbb{E}_3$ -space map

$$f_{p,n} : U[\alpha(p, n)] \rightarrow BGL_1(MU_{(p)}),$$

has its associated $\mathbb{E}_3$ - $MU_{(p)}$ -Thom spectrum $T(f_{p,n})$ is isomorphic to $\mathbb{Z}/p^n$.

Proof of Theorem 3.2 part (ii). Consider the (split if and only if $\alpha(p, n) = 1$) fiber sequence

$$SU_{(p)} \rightarrow U[\alpha(p, n)] \rightarrow S_{(p)}^1$$

of spaces. A triple looping and obstruction theory argument shows that this fiber sequence is one of $\mathbb{E}_3$ -spaces. Since $S_{(p)}^1$ is connected and the map $f_p : SU_{(p)} \rightarrow BGL_1(MU_{(p)})$ has associated $\mathbb{E}_3$ - $MU_{(p)}$ -Thom spectrum isomorphic to $\mathbb{Z}_{(p)}$, by the intermediate Thom spectrum theorem, we have the following commutative diagram

$$\begin{array}{ccccc} SU_{(p)} & \xrightarrow{i} & U[\alpha(p, n)] & \xrightarrow{\quad} & S^1 \\ & \searrow f_p & \downarrow f_{p,n} & & \downarrow \\ & & BGL_1(MU_{(p)}) & \longrightarrow & BGL_1(T(SU_{(p)})) \rightarrow BGL_1(MU_{(p)}) : \end{array}$$

of $\mathbb{E}_3$ -spaces. Note that $BGL_1(T(SU_{(p)} \rightarrow BGL_1(MU_{(p)})) = BGL_1(\mathbb{Z})_{(p)}$ so the square in the diagram can be rewritten in the following way:

$$\begin{array}{ccc} U_{(p)} & \xrightarrow{\quad} & S_{(p)}^1 \\ \downarrow f_{p,n} & & \downarrow \\ BGL_1(MU_{(p)}) & \longrightarrow & BGL_1(\mathbb{Z}_{(p)}) \end{array}$$

We have $T(f)$ is isomorphic as an MU -module to the colimit of the diagram

$$S^1 \rightarrow BGL_1(\mathbb{Z}_{(p)}).$$

which we've calculated to be $(1+p)^{\alpha(p,n)}$ in Lemma ???. Since $S_{(p)}^1 = B\mathbb{Z}_{(p)}$, $T(f)$ is isomorphic to the coinvariants of $\mathbb{Z}_{(p)}$ acting on $\mathbb{Z}_{(p)}$ where the action of 1 is by $(1+p)^{\alpha(p,n)}$. Since the p -adic valuation of $1 - (1+p)^{\alpha(p,n)}$ is n , we conclude that the $\mathbb{E}_3$ - $MU_{(p)}$ -Thom spectrum of $f_{p,n}$ is isomorphic to $\mathbb{Z}/p^n$. $\square$

Finally we see how to deduce Theorem 3.2 part (i) whose statement we recall. There an $\mathbb{E}_3$ -space map

$$\overline{f}_2 : U \rightarrow BGL_1(MU),$$

an integral (existing before 2-localization) analogue of f_2 , whose associated $\mathbb{E}_3$ - MU -Thom spectrum is isomorphic to $\mathbb{F}_2$.

Proof of Theorem 3.2 Part (i). The proof of Lemma 3.9 can be slightly modified to show that there exists an $\mathbb{E}_3$ -space map $\overline{f}_2 : U \rightarrow BGL_1(MU_{(p)})$ such that

$$U(1) \hookrightarrow U \xrightarrow{\overline{f}_2} BGL_1(MU_{(p)})$$

corresponds to the (unique) non-trivial element of $\pi_1(BGL_1(MU_{(p)})) \cong \mathbb{Z}^\times \cong \mathbb{F}_2$. Furthermore the proof of Theorem 3.2 part (ii) can be slightly modified to conclude Theorem 3.2 Part (i). $\square$

3.2. Nonexistence Results of Theorem A. In this subsection we prove parts (v) through (ix) of Theorem 3.2. We begin with parts (v), (vi), and (vii) whose proofs are observations of non-compatibility with power operations. We make frequent use of and Lemma 2.8 and Proposition 2.11 without reference.

We prove Theorem 3.2 part (v) whose statement we recall. For all primes p , there does not exist a p -complete $\mathbb{E}_4$ -space map

$$U[\alpha(p, 1)] \rightarrow \mathrm{BGL}_1(\mathrm{MU}_p^\wedge)$$

whose associated $\mathbb{E}_4$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{F}_p$.

Proof of Theorem 3.2 Part(v). It suffices to show there there does not exist an equivalence

$$g : \mathbb{F}_p[U_p^\wedge] \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}_p^\wedge} \mathbb{F}_p$$

of $\mathbb{E}_4$ - $\mathbb{F}_p$ -algebras as the existence of a p -complete $\mathbb{E}_4$ -space map

$$U[\alpha(p, 1)] \rightarrow \mathrm{BGL}_1(\mathrm{MU}_p^\wedge)$$

with associated $\mathbb{E}_4$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{F}_p$ would imply the existence of the equivalence g by a Thom isomorphism argument.

Note that g induces a homotopy ring map

$$\pi_*(g) : \Lambda_{\mathbb{F}_p}(\sigma c_0, \sigma c_1, \dots) \rightarrow \Lambda_{\mathbb{F}_p}(\sigma x_0, \sigma x_1, \dots).$$

with $|\sigma c_i| = |\sigma x_i|$ for all $i \geq 0$. Consider $\sigma c_0 = \sigma p \in \pi_1(\mathbb{F}_p[U_p^\wedge])$. By cite, for $p = 2$, the $\mathbb{E}_4$ -operation Q^4 applied to σc_0 is $\sigma c_2 \in \pi_5(\mathbb{F}_2[U_2^\wedge])$ mod units and decomposables. Likewise, for p odd, the $\mathbb{E}_4$ operation Q^2 applied to σc_0 is $\sigma c_{2p-2} \in \pi_{4p-3}(\mathbb{F}_2[U_2^\wedge])$ mod units and decomposables.. Regardless of the parity of the prime p , the $\mathbb{E}_4$ -Dyer-Lahof operation on σc_0 is non-zero. However the analogous $\mathbb{E}_4$ -operation on $\sigma x_0 \in \pi_1(\mathbb{F}_p \otimes_{\mathrm{MU}_p^\wedge} \mathbb{F}_p)$ vanishes. $\square$

We prove Theorem 3.2 part (vi) whose statement we recall. For all primes p and integers $n > 1$ other than $(p, n) = (2, 2)$, there does not exist a p -complete $\mathbb{E}_{2p+2}$ -space map

$$U[\alpha(p, n)] \rightarrow \mathrm{BGL}_1(\mathrm{MU}_p^\wedge)$$

whose associated $\mathbb{E}_{2p+2}$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{Z}/p^n$.

Proof of Theorem 3.2 Part(vi). It suffices to show there there does not exist an equivalence

$$g : \mathbb{Z}/p^n[U_p^\wedge] \rightarrow \mathbb{F}_p \otimes_{\mathrm{MU}_p^\wedge} \mathbb{Z}/p^n$$

of $\mathbb{E}_{2p+2}$ - $\mathbb{F}_p$ -algebras as the existence of a p -complete $\mathbb{E}_{2p+2}$ -space map

$$U[\alpha(p, n)] \rightarrow \mathrm{BGL}_1(\mathrm{MU}_p^\wedge)$$

with associated $\mathbb{E}_4$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{Z}/p^n$ would imply the existence of the equivalence g by a Thom isomorphism argument.

Note that g induces a homotopy ring map

$$\pi_*(g) : \Lambda_{\mathbb{F}_p}(\sigma c_0, \sigma c_1, \dots) \rightarrow \Lambda_{\mathbb{F}_p}(\sigma x_0, \sigma x_1, \dots).$$

with $|\sigma c_i| = |\sigma x_i|$ for all $i \geq 0$. Consider $\sigma c_{p-1} \in \pi_{2p-1}(\mathbb{F}_p[U_p^\wedge])$. By cite, for $p = 2$, the $\mathbb{E}_4$ -operation Q^8 applied to σc_1 is $\sigma c_5 \in \pi_{11}(\mathbb{F}_2[U_2^\wedge])$. Likewise, for p odd, the $\mathbb{E}_{2p+2}$ operation Q^{2p} applied to σc_{p-1} is $-\sigma c_{2p^2-p-2} \in \pi_{4p^2-2p-1}(\mathbb{F}_2[U_2^\wedge])$. Regardless of the parity of the prime p , the $\mathbb{E}_{2p+2}$ -Dyer-Lahof operation on σc_{p-1} is non-zero. However, the analogous $\mathbb{E}_{2p+2}$ -operation on $\sigma x_{p-1} \in \pi_1(\mathbb{F}_p \otimes_{\mathrm{MU}_p^\wedge} \mathbb{Z}/p^n)$ vanishes. $\square$

We prove Theorem 3.2 part (vii) whose statement we recall. For all primes p there does not exist a p -complete $\mathbb{E}_4$ -space map

$$SU \rightarrow \mathrm{BGL}_1(\mathrm{MU}_p^\wedge)$$

whose associated $\mathbb{E}_{2p+2}$ - $\mathrm{MU}_p^\wedge$ -Thom spectrum is isomorphic to $\mathbb{Z}_p^\wedge$.

Proof of Theorem 3.2 Part(vii). This can be proven by a slight variation of the above proof of part (vi) of Theorem 3.2. $\square$

Now that we've proven parts (v), (vi), and (vii) of Theorem 3.2, we move on to the proofs of parts (viii) and (ix) whose proofs differ from the proofs of parts (v), (vi), and (vii) in that they aren't completely explained by a power operation obstruction.

We prove Theorem 3.2 part (viii) whose statement we recall. There does not exist a $\mathbb{E}_2$ -space map

$$SU \rightarrow \mathrm{BGL}_1(\mathrm{MU})$$

whose associated $\mathbb{E}_2$ - MU -Thom spectrum is isomorphic to $\mathbb{Z}$.

Proof of Theorem 3.2 Part(viii). It suffices to show there does not exist an equivalence

$$g : \mathbb{Z}[\mathrm{SU}] \rightarrow \mathbb{Z} \otimes_{\mathrm{MU}} \mathbb{Z}$$

of $\mathbb{E}_2$ - $\mathbb{Z}$ -algebras as the existence of a $\mathbb{E}_2$ -space map

$$SU \rightarrow \mathrm{BGL}_1(\mathrm{MU})$$

with associated $\mathbb{E}_2$ - MU -Thom spectrum is isomorphic to $\mathbb{Z}$ would imply the existence of the equivalence g by a Thom isomorphism argument.

Thus we let g denote any such map of $\mathbb{E}_2$ - $\mathbb{Z}$ -algebras corresponding to an $\mathbb{E}_3$ -ring map MU to MU specified by coordinates $(c_1, c_2, \dots)$, and will show g is not an equivalence. Base changing along $\mathbb{Z} \rightarrow \mathbb{F}_5$ yields a map

$$\mathbb{F}_5 \otimes_{\mathbb{Z}} g : \mathbb{F}_5[\mathrm{SU}] \rightarrow \mathbb{F}_5 \otimes_{\mathrm{MU}} \mathbb{Z}$$

of $\mathbb{E}_2$ - $\mathbb{F}_5$ -algebras. Consider h the composition

$$h : \mathbb{F}_5 \otimes_{\mathrm{MU}} \mathbb{Z} \rightarrow \mathbb{F}_5 \otimes_{\mathrm{MU}} \mathbb{F}_5 \rightarrow \mathbb{F}_5 \otimes_{\mathbb{F}_5 \otimes_{\mathrm{MU}} \mathbb{F}_5} \mathbb{F}_5 \cong \mathbb{F}_5[\mathrm{SU}]$$

where the first map is induced by the base change $\mathbb{Z} \rightarrow \mathbb{F}_5$, the second map is induced by the Hurewicz $S \rightarrow \mathrm{MU}$, and the last equivalence follows from the Thom isomorphism $\mathbb{F}_5[\mathrm{MU}] \cong \mathbb{F}_5[\mathrm{BU}]$. Composing g and h give the map

$$h \circ g : \mathbb{F}_5[\mathrm{SU}] \rightarrow \mathbb{F}_5[\mathrm{SU}]$$

of $\mathbb{E}_2$ - $\mathbb{F}_5$ -algebras. We recall (Milnor-Nikov) that the map $\mathrm{MU} \rightarrow \mathbb{Z} \otimes \mathrm{MU}$ induced Hurewicz $S \rightarrow \mathbb{Z}$ gives the homotopy ring map

$$\mathbb{Z}[x_1, x_2, \dots] \rightarrow \mathbb{Z}[b_1, b_2, \dots]$$

with $|x_i| = |b_i| = 2$ for all $i > 0$ where x_i is sent to $m_i b_i$ mod decomposables where

$$m_i = \begin{cases} q & i+1 = q^n \text{ for some prime } q \\ 1 & \text{otherwise} \end{cases}$$

so that x_1 is mapped to $2b_1$ and x_9 is mapped to b_9 mod decomposables. We also recall from CITE that the $\mathbb{E}_2$ -Dyer Lashof operation

$$Q^2 : \pi_3(\mathbb{F}_5[\mathrm{SU}]) \rightarrow \pi_{19}(\mathbb{F}_5[\mathrm{SU}])$$

satisfied $Q^2(\sigma b_1) = \sigma b_9 \bmod \text{units and decomposables}$. Putting everything together gives by naturality that $h \circ g$ sends σb_9 to $2\sigma b_9 \bmod \text{units and decomposables}$ which allows us to conclude that the coordinate c_9 cannot be the generator of $\pi_{18}(\text{MU}) \bmod \text{units and decomposables}$. By Lemma 3.5 this allows us to conclude that g cannot be an equivalence. $\square$

We prove Theorem 3.2 part (ix) whose statement we recall. For all primes p there does not exist a p -complete $\mathbb{E}_2$ -space map

$$BBSU \rightarrow \text{BGL}_1(\text{MU}_p^\wedge)$$

whose associated $\mathbb{E}_2\text{-MU}_p^\wedge$ -Thom spectrum is isomorphic to $\text{ku}_p^\wedge$.

Proof of Theorem 3.2 Part (ix). It suffices to show there does not exist an equivalence

$$g : \text{ku}[BBSU] \rightarrow \text{ku}_p^\wedge \otimes_{\text{MU}_p^\wedge} \text{ku}_p^\wedge$$

of $\mathbb{E}_2\text{-ku}_p^\wedge$ -algebras as the existence of a p -complete $\mathbb{E}_2$ -space map

$$BBSU \rightarrow \text{BGL}_1(\text{MU}_p^\wedge)$$

with associated $\mathbb{E}_2\text{-MU}_p^\wedge$ -Thom spectrum is isomorphic to $\text{ku}_p^\wedge$ would imply the existence of the equivalence g by a Thom isomorphism argument.

It suffices to show there is no $\mathbb{E}_1\text{-}\mathbb{F}_p$ -algebra equivalence

$$\mathbb{F}_p[\text{BU}\langle 6 \rangle] \rightarrow \mathbb{F}_p \bigotimes_{\mathbb{F}_p \otimes_{\text{MU}_p^\wedge} \text{ku}_p^\wedge} \mathbb{F}_p$$

as this would follow from the existence of the $\mathbb{E}_2\text{-ku}_p^\wedge$ -algebra map g by base changing $\text{ku}_p^\wedge \rightarrow \mathbb{F}_p$ then applying the two-sided $\mathbb{F}_p$ -bar construction. By [AHS01], the divided power algebra associative multiplication structure on $\pi_*(\mathbb{F}_p[\text{BU}\langle 6 \rangle])$ has nilpotence, so it suffices to show that the associative multiplication on

$$\pi_* \left(\mathbb{F}_p \bigotimes_{\mathbb{F}_p \otimes_{\text{MU}_p^\wedge} \text{ku}_p^\wedge} \mathbb{F}_p \right)$$

does not have nilpotence. Consider the algebra base change $\text{ku}_p^\wedge \rightarrow \mathbb{F}_p$ and the induced base change

$$\mathbb{F}_p \otimes_{\text{MU}_p^\wedge} \text{ku}_p^\wedge \rightarrow \mathbb{F}_p \otimes_{\text{MU}_p^\wedge} \mathbb{F}_p$$

which itself induces the map

$$\mathbb{F}_p \bigotimes_{\mathbb{F}_p \otimes_{\text{MU}_p^\wedge} \text{ku}_p^\wedge} \mathbb{F}_p \rightarrow \mathbb{F}_p \bigotimes_{\mathbb{F}_p \otimes_{\text{MU}} \mathbb{F}_p} \mathbb{F}_p \cong \text{THH}(\mathbb{F}_p / \text{MU})$$

of $\mathbb{E}_1\text{-}\mathbb{F}_p$ -algebras. The multiplication on $\text{THH}_*(\mathbb{F}_p / \text{MU})$ is polynomial (hahn wilson), and a Bar spectral sequence computation allows us to see that the induced map is injective on homotopy ring classes therefore cannot have nilpotence. $\square$

4. TOPOLOGICAL HOCHSCHILD HOMOLOGY AND THEOREM B

We begin by recalling the definition of conjugate stack.

Definition 4.1. *Let R be a connective $\mathbb{E}_3$ -MU-algebra such that $\mathrm{THH}(R/\mathrm{MU})$ is even, and define conjugate stack R^{conj} of R denote the graded quotient stack associated to the graded groupoid scheme*

$$\mathrm{Spec}(\mathrm{THH}_{2*}(R/\mathrm{MU})) \xleftarrow[\eta_R]{\eta_L} \mathrm{Spec}(\mathrm{THH}_{2*}(R/\mathrm{MU}^{\otimes 2})).$$

where η_L and η_R denote the maps induced by the left and right units of the Hopf algebroid $(\mathrm{MU}, \mathrm{MU} \otimes \mathrm{MU})$.

Theorem 4.2 (Theorem B). *(i) Let R be a connective $\mathbb{E}_3$ -MU-algebra such that $\mathrm{THH}(R/\mathrm{MU})$ is even. Then R^{conj} satisfies the following properties. Firstly, there is an equivalence of spectra*

$$\mathrm{gr}_{\mathrm{ev}}^t \mathrm{THH}(R) \simeq R\Gamma(R^{\mathrm{conj}}, \mathcal{O}\{t\})[2t]$$

for $\mathrm{fil}_{\mathrm{ev}}^*$ denoting Pstragowski's even filtration [Pst23, Theorem 1.3]. Furthermore is a strongly convergent multiplicative spectral sequence of signature

$$H^s(R^{\mathrm{conj}}, \mathcal{O}\{t\}) \Rightarrow \pi_{2t-s} \mathrm{THH}(R).$$

Lastly, if R is furthermore a discrete p -complete quasisyntomic commutative ring such that $\mathrm{THH}(R/\mathrm{MU})$ has bounded p -power torsion then there is an isomorphism of fqc stacks

$$\widehat{R^{\mathrm{conj}}}_p \cong (\mathrm{Spf}(R)^{\mathcal{N}})_{s=0}$$

between the formal p -completion of R^{conj} and the zero fiber $(\mathrm{Spf}(R)^{\mathcal{M}})_{s=0}$ of the Nygaard filtered prismaticization of $\mathrm{Spf}(R)$, the familiar definition of conjugate stack of R .

We record the following conjugate stack calculations of $R = \mathbb{Z}/p^n$ and $R = \mathrm{BP}\langle n \rangle$.

(ii) Let p be prime and $n > 0$ an integer such that $(p, n) \neq (2, 2)$. Then

$$(\mathbb{Z}/p^n)^{\mathrm{conj}} = \begin{cases} \mathbb{G}_{a, \mathbb{F}_p} / \mathbb{G}_m & n = 1 \\ \mathbb{G}_{a, \mathbb{Z}/p^n}^{\sharp} \times [\mathbb{G}_{a, \mathbb{Z}/p^n} / \mathbb{G}_{a, \mathbb{Z}/p^n}^{\sharp}] / \mathbb{G}_m & n > 1 \end{cases}$$

as affine group schemes over $\mathrm{Spec}(\mathbb{Z}/p^n) / \mathbb{G}_m$ where (for $n > 1$) the quotient is associated to the divided-power envelope $\mathbb{G}_{a, \mathbb{Z}/p^n}^{\sharp} \rightarrow \mathbb{G}_{a, \mathbb{Z}/p^n}$.

(iii) Let $(\mathcal{X}_{\mathrm{BP}\langle n \rangle})_{p=\dots=v_n}$ denote the mod $(p, v_1, \dots, v_n)$ reduction

$$\mathrm{BP}\langle n \rangle^{\mathrm{conj}} \times_{\mathrm{Spec}(\mathbb{Z}_p[v_1, \dots, v_n]) / \mathbb{G}_m} B\mathbb{G}_{m, \mathbb{F}_p}$$

of $\mathrm{BP}\langle n \rangle^{\mathrm{conj}}$. We have

$$\mathrm{BP}\langle n \rangle_{p=\dots=v_n=0}^{\mathrm{conj}} = [F_*^{m+1} W_{\mathbb{Z}_p} / F_* W_{\mathbb{Z}_p}] / \mathbb{G}_m = [\mathbb{G}_a / W[F^n]] \mathbb{G}_m$$

as affine group schemes over $\mathrm{Spec}(\mathbb{Z}_p[v_1, \dots, v_n]) / \mathbb{G}_m$ where the quotient is associated to $pF^n : F_* W_{\mathbb{F}_p} \rightarrow F_*^{n+1} W_{\mathbb{F}_p}$.

4.1. Bialgebraic Structure THH of Generalized Thom Spectra. We recall the THH-of-Thom spectra formula in Proposition 4.3.

Proposition 4.3 ([KN], Proposition 4.7). *Let $f : X \rightarrow BGL_1(\mathbb{R})$ be an $\mathbb{E}_n$ -map of spaces for $n \geq 2$, and suppose that the $\mathbb{E}_n$ - R -algebra structure of $T(f)$ extends to an $\mathbb{E}_{n+1}$ - R -algebra structure not necessarily induced from a space level structure on X . Then there is an equivalence*

$$\mathrm{THH}(T(f)/\mathbb{R}) \cong T(f)[BX]$$

of $\mathbb{E}_{n-1}$ - R -algebras.

Corollary 4.4. *Using generalized Thom spectra equivalences of Theorem A and Proposition 4.3, we have the following equivalences*

(1) *There is an equivalence*

$$\mathrm{THH}(\mathbb{F}_2/\mathrm{MU}) \cong \mathbb{F}_2[\mathrm{BU}]$$

of $\mathbb{E}_2$ - MU -algebras.

(2) *For each prime p and $n > 0$ (other than $(p, n) = (2, 2)$) there is an equivalence*

$$\mathrm{THH}((\mathbb{Z}/p^n)/\mathrm{MU}) \cong (\mathbb{Z}/p^n)[\mathrm{BU}[\alpha(p, n)]]$$

of $\mathbb{E}_2$ - MU -algebras.

(3) *For each prime p , there is an equivalence*

$$\mathrm{THH}(\mathbb{Z}_{(p)}/\mathrm{MU}_{(p)}) \cong \mathbb{Z}_{(p)}[\mathrm{BSU}]$$

of $\mathbb{E}_2$ - $\mathrm{MU}_{(p)}$ -algebras

Proof. The only thing needing justification is the $\mathbb{Z}/p^n$ case which follows from being p -torsion and localization being smashing. $\square$

We will generalize Proposition 4.3 to bialgebraic structures. We begin by introducing $\mathbb{E}_n$ -coalgebras following [Bea23, Section 3]. and (i, j) -bialgebras following [Bea23, Section 3.1] The ∞ -category of $\mathbb{E}_n$ -coalgebras $\mathrm{CoAlg}_{\mathbb{E}_n}(\mathcal{C})$ in an $\mathbb{E}_n$ -monoidal ∞ -category $\mathcal{C}$ is defined to be $(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$.

Lemma 4.5 (Proposition 4.9 and Corollary 4.14 in [Bea23]). *Fix $n > 0$. For R an $\mathbb{E}_{n+1}$ -ring and X an $\mathbb{E}_n$ -space, let $f : X \rightarrow \mathrm{Pic}(R)$ denote the constant $\mathbb{E}_n$ -space map. Then the R -Thom spectrum $T(f) \cong R[X]$ is a counital $\mathbb{E}_n$ -coalgebra in Mod_R with coproduct*

$$T(f) \cong R[X] \xrightarrow{R[\Delta_X]} R[X \times X] \cong R[X] \otimes_R R[X] \cong T(f) \otimes_R T(f)$$

and counit

$$T(f) \cong R[X] \xrightarrow{R[X \rightarrow *]} R.$$

Furthermore $T(f)$ may be regarded as an (i, j) -bialgebra in Mod_R for all $i + j \leq \min(n, m)$ with $i, j \geq 0$. Finally, $T(f)$ may be regarded as a counital (right) $R[X]$ -comodule in Mod_R which coaction $\Delta_{T(f)}$ given by the Thom diagonal:

$$T(f) \cong R[X] \xrightarrow{R[\Delta_X]} R[X \times X] \cong R[X] \otimes_R R[X] \cong T(f) \otimes_R T(f).$$

Recall that there exists a category of augmented of augmented $\mathbb{E}_n$ - R -modules, that is, the overcategory of $\mathbb{E}_n$ - R -modules over R .

Example 4.6. *Fix $n > 0$. For R an $\mathbb{E}_{n+1}$ -ring and X an $\mathbb{E}_n$ -space, let $f : X \rightarrow \mathrm{Pic}(R)$ denote an $\mathbb{E}_n$ -space map, and let $T(f)$ denote its corresponding $\mathbb{E}_n$ - R -Thom spectrum.*

- (1) The $\mathbb{E}_n$ - R -algebra $T(f)[X]$ admits an $\mathbb{E}_n$ - $T(f)$ -algebra structure and admits an augmented $\mathbb{E}_n$ - $T(f)$ -algebra structure via the map

$$T(f)[X] \xrightarrow{T(f)[*X]} T(f).$$

which is the counit ϵ

- (2) The $\mathbb{E}_n$ - R -algebra $T(f) \otimes_R T(f)$ admits an $\mathbb{E}_n$ - $T(f)$ -algebra structure and further admits an augmented $\mathbb{E}_n$ - $T(f)$ -algebra structure via the multiplication

$$\mu_{T(f)} : T(f) \otimes_R T(f) \rightarrow T(f)$$

on $T(f)$.

In the following Lemma, we refer to the augmentation structures studied in Examples 4.6.

Lemma 4.7. *Fix $n > 0$. For R an $\mathbb{E}_{n+1}$ -ring and X an $\mathbb{E}_n$ -space, let $f : X \rightarrow \text{Pic}(R)$ denote an $\mathbb{E}_n$ -space map, and let $T(f)$ denote its corresponding $\mathbb{E}_n$ - R -Thom spectrum. Then the Thom isomorphism*

$$\phi_f : T(f) \otimes_R T(f) \cong T(f)[X]$$

respects the $T(f)$ -augmentations therefore is an isomorphism of augmented $\mathbb{E}_n$ - $T(f)$ -algebras.

Proof. This follows from the coherent factorization of the Thom diagonal [Bea23] and counitality as in Lemma 4.5. $\square$

Now we state the bialgebraic “THH of Thom spectra formula”.

Proposition 4.8 (Proposition ??). *Fix $n > 1$ and R an $\mathbb{E}_{n+1}$ -ring. Let $f : X \rightarrow \text{Pic}(R)$ denote a pointed $\mathbb{E}_n$ -space map. Let $T(f)$ denote its corresponding $\mathbb{E}_n$ - R -Thom spectrum and suppose that $T(f)$ admits an $\mathbb{E}_{n+1}$ - R -algebra structure not necessarily refining its $\mathbb{E}_n$ - R -algebra structure from being an $\mathbb{E}_n$ - R -Thom spectrum of f . Then there exists an equivalence*

$$\text{THH}(T(f)/R) \simeq T(f)[BX]$$

of counital $\mathbb{E}_1$ -coalgebras in augmented $\mathbb{E}_{n-1}$ - $T(f)$ -algebras.

Proof. From Lemma 4.7, we recall that the Thom isomorphism gives an equivalence

$$\phi_f : T(f) \otimes_R T(f) \simeq T(f)[X]$$

of augmented $\mathbb{E}_n$ - $T(f)$ -algebras. It suffices to argue the bar construction can be viewed as a functor from augmented $\mathbb{E}_n$ - $T(f)$ -algebras to $\mathbb{E}_1$ -coalgebras in augmented $\mathbb{E}_{n-1}$ - $T(f)$ -algebras so that applying the bar construction yields the isomorphism

$$\begin{aligned} B(T(f), \phi_f, T(f)) : \text{THH}(T(f)/R) &\cong T(f)_{T(f) \otimes_R T(f)} T(f) \\ &\rightarrow T(f) \otimes_{T(f)[X]} T(f) \\ &\cong T(f)[BX]. \end{aligned}$$

Indeed, by Dunn additivity we have that

$$\text{Alg}_{\mathbb{E}_n}^{\text{aug}}(\text{Mod}_R) = \text{Alg}_{E_1}(\text{Alg}_{\mathbb{E}_{n-1}}^{\text{aug}}(\text{Mod}_R)),$$

and the bar construction can be thought of as a functor

$$\text{Bar} : \text{Alg}_{\mathbb{E}_1}(\mathcal{C}) \rightarrow \text{CoAlg}_{\mathbb{E}_1}(\mathcal{C})$$

for $\mathcal{C} = \text{Alg}_{\mathbb{E}_{n-1}}^{\text{aug}}(\text{Mod}_R)$. $\square$

Proposition 4.9. *Fix $n > 2$, $\mathbb{E}_{n+1}$ -ring R , and an $\mathbb{E}_n$ -map $f : X \rightarrow \mathrm{Pic}(R)$. Let $T(f)$ denote the $\mathbb{E}_n$ -R-Thom spectrum of f . Suppose $T(f)_*[BX]$ is flat over $\pi_*(T(f))$. Then there exists an equivalence*

$$\mathrm{THH}_*(T(f)/R) \rightarrow T(f)_*[BX]$$

of graded bicommutative $T(f)_$ -Hopf algebras.*

Proof. By Lemma ??, we have an equivalence

$$\mathrm{THH}(T(f)/R) \cong R[BX]$$

of counital $\mathbb{E}_1$ -coalgebras in augmented $\mathbb{E}_{n-1}$ - $T(f)$ -algebras. On homotopy rings this induces an equivalence of graded commutative $T(f)_*$ -algebras and graded cocommutative $T(f)_*$ -coalgebras. By flatness of $T(f)_*[BX]$ over $\pi_*(T(f))$, we obtain the desired graded bicommutative $T(f)_*$ -bialgebra equivalence. Finally this automatically upgrades to an equivalence graded bicommutative $T(f)_*$ -Hopf algebras with antipode give by inversion on the group structure of BX . $\square$

Finally we state the following calculation of $\mathrm{THH}_*(\mathbb{F}_p/\mathrm{MU})$ from Corollary 4.4 in Corollary 4.13. To do so, we recall from [AHS01, Proposition 4.1] the equivalence of $\mathrm{Spec}(H_*(\mathrm{BU}, \mathbb{F}_p))$ and $\mathrm{C}^1(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$ as graded group schemes over $\mathbb{F}_p$. This induces an equivalence of $H_*(\mathrm{BU}, \mathbb{F}_p)$ with the coordinate ring $\mathcal{O}_{\mathrm{C}^1(\widehat{\mathbb{G}}_a, \mathbb{G}_m)}$ of graded $\mathbb{F}_p$ -bialgebras.

Proposition 4.10. *We have*

$$\mathrm{THH}_*(\mathbb{F}_p/\mathrm{MU}) \cong \pi_*(\mathbb{F}_p[\mathrm{BU}])$$

as bicommutative $\mathbb{F}_p$ -Hopf algebras,

$$\mathrm{THH}_*(\mathbb{Z}/p^n/\mathrm{MU}) \cong \pi_*((\mathbb{Z}/p^n)[B(U[\alpha(p, n)])])$$

as bicommutative $\mathbb{Z}/p^n$ -Hopf algebras and

$$\mathrm{THH}_*(\mathbb{Z}_{(p)}/\mathrm{MU}) \cong \pi_*(\mathbb{Z}_{(p)}[\mathrm{BSU}])$$

as bicommutative $\mathbb{Z}_{(p)}$ -Hopf algebras.

4.2. Conjugate Stacks. We now prove Theorem B part (i) whose statement we recall. Let R be a connective $\mathbb{E}_3$ -MU-algebra such that $\mathrm{THH}(R/\mathrm{MU})$ is even. Thus there exists a conjugate stack R^{conj} . Then R^{conj} satisfies the following properties. For one, Then

$$\mathrm{gr}_{\mathrm{ev}}^t \mathrm{THH}(R) \cong R\Gamma(R^{\mathrm{conj}}, \mathcal{O}\{t\})[2t].$$

Furthermore there is a strongly convergent spectral sequence of signature

$$H^s(R^{\mathrm{conj}}, \mathcal{O}\{t\}) \Rightarrow \pi_{2t-s} \mathrm{THH}(R).$$

Lastly, if R is furthermore a discrete p -complete quasisyntomic commutative ring such that $\mathrm{THH}(R/\mathrm{MU})$ has bounded p -power torsion then there is an isomorphism of fpc stacks

$$\widehat{R^{\mathrm{conj}}}_p \cong (\mathrm{Spf}(R)^{\mathcal{N}})_{s=0}$$

between the formal p -completion of R^{conj} and the zero fiber $(\mathrm{Spf}(R)^{\mathcal{M}})_{s=0}$ of the Nygaard filtered prismaticization of $\mathrm{Spf}(R)$, the familiar definition of conjugate stack of R .

Proof of Theorem 4.2, Part (i). We first show that

$$\mathrm{gr}_{\mathrm{ev}}^t \mathrm{THH}(R) \cong R\Gamma(R^{\mathrm{conj}}, \mathcal{O}\{t\})[2t].$$

Since the augmentation $\mathrm{THH}(\mathrm{MU}) \rightarrow \mathrm{MU}$ is faithfully evenly flat, base change shows that $\mathrm{THH}(R) \rightarrow \mathrm{THH}(R/\mathrm{MU})$ is faithfully evenly flat. By faithfully flat descent the natural comparison

$$\mathrm{fil}_{\mathrm{ev}}(\mathrm{THH}(R)) \rightarrow \lim_{n \in \Delta} \mathrm{fil}_{\mathrm{ev}}(\mathrm{THH}(R/\mathrm{MU}^{\otimes n+1}))$$

of filtered spectra is an equivalence asfter Postnikov completion. By the assumption of connectivity of R we learn this natural comparison of filtered spectra is an equivalence before Postnikov completion. For each n we have that $\mathrm{THH}(R/\mathrm{MU}^{\otimes n+1})$ is even (by assumption that $\mathrm{THH}(R/\mathrm{MU})$ is even) thus

$$\mathrm{fil}_{\mathrm{ev}}(\mathrm{THH}(R/\mathrm{MU}^{\otimes n+1})) \simeq \tau_{\geq 2*}(\mathrm{THH}(R/\mathrm{MU}^{\otimes n+1}))$$

as filtered spectra. On the other hand by fpqc descent we have

$$R\Gamma(R^{\mathrm{conj}}, \mathcal{O}\{t\}) \simeq \lim_{n \in \Delta} H(\pi_{2t}(\mathrm{THH}(R/\mathrm{MU}) \otimes_{\mathrm{THH}(R)} \mathrm{THH}(R/\mathrm{MU}))^{\otimes_{\mathrm{THH}(R/\mathrm{MU})} n}),$$

as (derived) $\mathbb{Z}$ -modules. So the argument finishes by observing the equivalence

$$\mathrm{THH}(R/\mathrm{MU}^{\otimes n+1}) \simeq \mathrm{THH}(R/\mathrm{MU}) \otimes_{\mathrm{THH}(R)} \mathrm{THH}(R/\mathrm{MU})^{\otimes_{\mathrm{THH}(R/\mathrm{MU})} n}$$

as $\mathbb{E}_2$ -rings.

The existence of the strongly convergent spectral sequence of signature

$$H^s(R^{\mathrm{conj}}, \mathcal{O}\{t\}) \Rightarrow \pi_{2t-s} \mathrm{THH}(R).$$

follows from [Pst23, Theorem 1.8].

Finally we show the claim about when R is furthermore a discrete p -complete quasisyntomic commutative ring such that $\mathrm{THH}(R/\mathrm{MU})$ has bounded p -power torsion. By assumption that R is a discrete commutative ring, R admits an $\mathbb{E}_{\mathrm{inf}}\text{-}\mathrm{MU}$ -algebra structure. There exists a quasiregular semiperfectoid S and cover $R \rightarrow S$ in the quasisyntomic topology as explained in [HRW22, Proof of Theorem 5.0.3]. Both of the induced maps $\mathrm{THH}(R; \mathbb{Z}_p^\wedge) \rightarrow \mathrm{THH}(R/\mathrm{MU})_p^\wedge$ and $\mathrm{THH}(R; \mathbb{Z}_p^\wedge) \rightarrow \mathrm{THH}(S; \mathbb{Z}_p^\wedge)$ are p -completely eff. By assumption $\mathrm{THH}(R/\mathrm{MU})_p^\wedge$ has bounded p -power torsion, and $\mathrm{THH}(S; \mathbb{Z}_p^\wedge)$ has bounded p -power torsion due to S being quasiregular semiperfectoid. Thus two faithfully flat descent statements show that $\mathrm{fil}_{\mathrm{ev}} \mathrm{THH}(R)_p^\wedge$ the natural comparison map of filtered $\mathbb{E}_\infty$ -rings into the limit

$$\lim_{(n,m) \in \Delta \times \Delta} \tau_{\geq *} \mathrm{THH}(R/\mathrm{MU}^{\otimes n+1}) \otimes_{\mathrm{THH}(R; \mathbb{Z}_p^\wedge)} \mathrm{THH}(S^{\otimes m+1}; \mathbb{Z}_p^\wedge)$$

where, as the notation suggests, the two limits commute so that the order in which they are computed doesn't change the resulting limit. By [BMS19, Theorem 1.12] we have the equivalence of graded (discrete) rings

$$\pi_{2*} \mathrm{THH}(S; \mathbb{Z}_p^\wedge) \simeq \bigoplus_{i \geq 0} \mathrm{gr}_{\mathcal{N}^i} \widehat{\Delta}_S \{i\}$$

where $\{i\}$ denotes the Breuil-Kisin twist. By [BL22, Definition 5.8.5], we can put Δ_S in place of $\widehat{\Delta}_S$ due to being Nygaard complete and therefore obtain the equivalence

$$[\mathrm{Spf}(\pi_*(\mathrm{THH}(S; \mathbb{Z}_p^\wedge)))/\mathbb{G}_m] \simeq (\mathrm{Spf}(R)^{\mathcal{N}})_{s=0}$$

given by the Rees description of prismaticization [Bha22, Section 5.5]. Thus we obtain $(\mathrm{Spf}(R)^{\mathcal{N}})_{s=0}$ by stackifying the $\Delta \times \Delta$ -indexed diagram of rings to obtain a $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ -indexed diagram of schemes and computing the two colimits in one of the two possible orders. Doing so in the other way gives $\widehat{R^{\mathrm{conj}}}_p$ as desired. $\square$

4.3. Conjugate Stack Calculations of Theorem B.

Notation 4.11. We establish notation following [Dev25, Example 7.6.5]. Let $\mathbf{W}$ denote the integral Witt group scheme, and let $\mathbf{W}(-*)$ denote the graded lift of $\mathbf{W}$ where the i -th coordinate has weight i . Likewise, we let $\mathbf{W}$ denote the p -typical group scheme and let $\mathbf{W}(-*)$ denote the graded lift of $\mathbf{W}$ where the p^i -th coordinate has weight p^i .

We let $\widehat{\mathbb{G}}_a$ denote the additive formal group over $\mathbb{F}_p$, $\mathbb{G}_m$ denote the multiplicative group over $\mathbb{F}_p$, $\underline{C}^k(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$ denote the group schemes introduced in [AHS01, Section 3].

For $0 \leq k \leq 3$ that by [AHS01] we have identifications $\mathrm{Spec}(\mathbb{F}_p[\mathrm{BU}\langle 2k \rangle]) \cong C^k(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$ of group schemes.

Lemma 4.12. Then the maps of graded affine commutative group schemes over $\mathrm{Spec}(\mathbb{F}_p)$

$$\begin{aligned} \mathrm{Spec}(\mathrm{THH}_*(\mathbb{F}_p / \mathrm{MU}_{(p)})) &\leftarrow \mathrm{Spec}(\pi_* R[\mathrm{BU}]) \\ &\xrightarrow{f_1} C^1_{\mathbb{F}_p}(\widehat{\mathbb{G}}_a, \mathbb{G}_m) \\ &\xleftarrow{\exp} \mathbf{W}_{\mathbb{F}_p}(-*) \end{aligned}$$

are equivalences. Note f_1 denotes the cocycle classification map and $\exp$ denotes the Artin-Hasse exponential. Furthermore for all (p, n) other than $(p, n) = (2, 2)$ we have

$$\begin{aligned} \mathrm{Spec}(\mathrm{THH}_*(\mathbb{Z}/p^n / \mathrm{MU}_{(p)})) &\leftarrow \mathrm{Spec}(\pi_* \mathbb{Z}/p^n [B(U(\alpha(p, n)))] \\ &\xrightarrow{f_1} C^1_{\mathbb{F}_p}(\widehat{\mathbb{G}}_a, \mathbb{G}_m) \times \mathbb{G}^{\sharp}_{a, \mathbb{Z}/p^n} \\ &\leftarrow \mathbf{W}_{\mathbb{F}_p}(-*) \times \mathbb{G}^{\sharp}_{a, \mathbb{Z}/p^n} \end{aligned}$$

and finally

$$\begin{aligned} \mathrm{Spec}(\mathrm{THH}_*(\mathbb{Z}_{(p)} / \mathrm{MU}_{(p)})) &\leftarrow \mathrm{Spec}((\mathbb{Z}_{(p)})_* [\mathrm{BSU}]) \\ &\xrightarrow{f_1} C^2(\widehat{\mathbb{G}}_a, \mathbb{G}_m) \times \mathrm{Spec}(\mathbb{Z}_{(p)}) \\ &\leftarrow F_* \mathbf{W}(-*)_{\mathbb{Z}/p^n} \end{aligned}$$

Proof. Our work of the bialgebraic THH of Thom spectra proves the first arrows in these equivalences. The middle arrows follow from [AHS01]. The last arrows are classical. $\square$

Corollary 4.13. We have equalities

$$\mathrm{THH}_*(\mathbb{F}_p / \mathrm{MU}) \cong H_*(\mathrm{BU}, \mathbb{F}_p) \cong \mathcal{O}_{C^1(\widehat{\mathbb{G}}_a, \mathbb{G}_m)}$$

of graded commutative $\mathbb{F}_p$ -bialgebras claimed in [Dev25, pg. 269]

Lemma 4.14. *The Quillen idempotent $\mathrm{MU} \rightarrow \mathrm{BP} \rightarrow \mathrm{MU}$ induces on*

$$\mathrm{Spec}(\mathrm{THH}_*(\mathbb{Z}_{(p)}/\mathrm{MU})) \cong \mathrm{Spec}(\pi_*(\mathbb{Z}_{(p)} \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{THH}(\mathrm{MU}/\mathrm{MU}^{\otimes 2})) \cong F_*W \times \prod_{(n,p)=1} W$$

the projection

$$F_*W \times \prod_{(n,p)=1} W \xrightarrow{id_{F_*W} \times 0} F_*W \times \prod_{(n,p)=1} W$$

*onto F_*W .*

Proof. This follows from tracking where the elements $\sigma^2 v_i$ are sent and where the other elements $\sigma^2 x_i$ are sent by the maps of Lemma 4.12. $\square$

Corollary 4.15. *The Cartier idempotent $\mathbf{W} \rightarrow \mathbf{W}$ splitting off W determines equivalences*

$$\begin{aligned} \mathrm{Spec}(\mathrm{THH}_*(\mathbb{F}_p/\mathrm{BP})) &\xrightarrow{f_1} C_{\mathbb{F}_p, \mathrm{typ}}^1(\widehat{\mathbb{G}}_a, \mathbb{G}_m) \\ &\leftarrow W_{\mathbb{F}_p}(-*) \end{aligned}$$

of graded affine group schemes over $\mathrm{Spec}(\mathbb{F}_p)$. It also determines equivalences

$$\begin{aligned} \mathrm{Spec}(\mathrm{THH}_*((\mathbb{Z}_{(p)})/\mathrm{BP})) &\xrightarrow{f_2} C_{\mathbb{Z}_{(p)}, \mathrm{typ}}^2(\widehat{\mathbb{G}}_a, \mathbb{G}_m) \\ &\leftarrow F_*W_{\mathbb{Z}_{(p)}}(-*) \end{aligned}$$

of graded affine group schemes over $\mathrm{Spec}(\mathbb{Z}_{(p)})$.

We call $C_{\mathbb{F}_p, \mathrm{typ}}^1(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$ and $C_{\mathbb{Z}_{(p)}, \mathrm{typ}}^2(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$ the group schemes cut out of $C^1(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$ and $C^2(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$, respectively.

Lemma 4.16. *The map*

$$\eta_R^{\mathrm{BP}} : \mathrm{THH}(\mathbb{Z}_{(p)}/\mathrm{BP}) \rightarrow \mathrm{THH}(\mathbb{Z}_{(p)}/\mathrm{BP}^{\otimes 2}) \cong \mathrm{THH}(\mathbb{Z}_{(p)}/\mathrm{BP}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \otimes_{\mathrm{THH}(\mathrm{BP})} \mathrm{BP}$$

satisfies

$$\eta_R^{\mathrm{BP}}(\sigma^2 v_i) = \sigma^2 v_i \otimes 1 + p \otimes \sigma^2 t_i$$

for all $i > 0$.

Proof. We write $\mathrm{BP}_* \mathrm{BP} \cong \mathrm{BP}[t_1, t_2, \dots]$ with $|t_i| = 2p^i - 2$. Recall, say from [Rav86], that

$$p\ell_n = \sum_{i=1}^{n-1} \ell_i v_{n-i}^{p^i}$$

and

$$\eta_R^{\mathrm{BP}}(\ell_n) = \sum_{i=0}^n \ell_i t_{n-i}^{p^n}$$

so that $\eta_R(v_i) \equiv v_i + pt_i \pmod{I^2}$ for I denoting the ideal $\ker(\mathrm{BP}_* \mathrm{BP} \rightarrow \mathbb{Z}_{(p)})$. The result follows from properties of suspension [Lee26, Section 2] and that I/I^2 is torsion-free. $\square$

Proposition 4.17. *The Hopf algebroid*

$$\mathrm{THH}_{2*}(\mathbb{F}_p/\mathrm{BP}) \begin{matrix} \xrightarrow{\eta_L} \\ \xleftarrow{\eta_R} \end{matrix} \mathrm{THH}_{2*}(\mathbb{F}_p/\mathrm{BP}^{\otimes 2})$$

gives the groupoid scheme

$$W \begin{matrix} \xleftarrow{w_1 \leftarrow (w_1, w_2)} \\ \xrightarrow{w_1 + V(w_2) \leftarrow (w_1, w_2)} \end{matrix} W \times F_* W$$

so that η_L is projection and η_R is $(V \otimes \mathrm{id})\Delta_W$ for Δ_W denoting the coproduct.

Proof. This follows from Lemma 4.16. □

At this point we've proven Theorem B part (ii) for $\mathrm{BP}\langle -1 \rangle = \mathbb{F}_p$.

Corollary 4.18 (Corollary 1.3). *We have*

$$F_p^{\mathrm{conj}} \cong \mathbb{G}_a / \mathbb{G}_m$$

as stacks over $\mathrm{Spec}(\mathbb{F}_p)/\mathbb{G}_m$ and that

$$\bigoplus_{t \in \mathbb{Z}} R\Gamma(\mathbb{F}_p^{\mathrm{conj}}, \mathcal{O}\{t\})[2t] \cong \mathbb{F}_p[x],$$

with $|x| = 2$, as graded $\mathbb{F}_p$ -modules. Thus the fact that $\mathbb{F}_p^{\mathrm{conj}} \cong \mathbb{G}_a / \mathbb{G}_m$ as stacks over $\mathrm{Spec}(\mathbb{F}_p)/\mathbb{G}_m$ is a “stacky” version of Bökstedt periodicity.

Lemma 4.19. *We have*

$$[\mathbb{G}_a / W[F^n]] \cong [F_*^n W / F_* W]$$

of quotient stacks.

Proof. We copy the proof of [Dri24, Proposition 3.5.1] but for F^m for m a positive integer in place of F . By the exactness of the sequence

$$0 \rightarrow F_* W \xrightarrow{V} W \xrightarrow{w_0} \mathbb{G}_a \rightarrow 0$$

we get $\mathbb{G}_a = [W/F_* W]$ thus

$$[G_a/W[F^m]] \cong [[W/F_* W]/W[F^m]].$$

Taking quotient stacks is associative for commuting actions. In particular, the Verschiebung action of $F_* W$ on W commutes with shift action of $W[F^n]$ on W therefore

$$[[W/F_* W]/W[F^m]] \cong [[W/W[F^n]]/F_* W].$$

Finally, the exactness of the sequence

$$0 \rightarrow W[F^m] \rightarrow VW \rightarrow F^n F_* W \rightarrow 0$$

gives $F_{ast}^n W \cong [W/W[F^m]]$ therefore

$$[[W/W[F^n]]/F_* W] \cong [F_*^n W / F_* W]$$

as desired. Note $F^n V = pF^{n-1}$ because of $FV = p$. □

Lemma 4.20. *Let A denote a graded $\mathbb{F}_p$ -Hopf subalgebra of W . the same dimension as $\mathbb{F}_p$ -module in each graded piece as the polynomial $\mathbb{F}_p$ -algebra $\mathbb{F}[x_n, x_{n+1}, x_{n+2}, \dots]$ with $\deg(i) = p^i$ and n fixed then A is isomorphic to $\mathcal{O}_{F_*^n}$ as a graded $\mathbb{F}_p$ -Hopf subalgebra of $\mathcal{O}_W$.*

Proof. Since A is $\mathbb{F}_p$ -Hopf subalgebra of W there exists a $\mathbb{F}_p$ -Hopf algebra map $\mathcal{O}_{W^\vee} \rightarrow A^\wedge$ which is the same as the $\mathbb{F}_p$ -Hopf algebra map $\mathcal{O}_W \rightarrow A^\wedge$ because W is self-dual. By assumption of the dimensions of the graded pieces of A , $W \rightarrow A^\wedge$ factors through $\mathcal{O}W/I \rightarrow A^\wedge$ for $I = (x_1, \dots, x_{n-1})$. The resulting map $\mathcal{O}/I \rightarrow A^\wedge$ is an isomorphism in each graded part therefore is an isomorphism of $\mathbb{F}_p$ -Hopf algebras. This concludes the proof of the uniqueness of the graded $\mathbb{F}_p$ -Hopf subalgebra structure of A , and there exists such an A given by $\mathcal{O}_{F_*^n}$. $\square$

Corollary 4.21. *The graded $\mathbb{F}_p$ -Hopf subalgebra $\mathrm{THH}_*(\mathrm{BP}\langle n \rangle / \mathrm{BP})$ of $\mathrm{THH}_*(\mathbb{F}_p / \mathrm{BP})$ induces the faithfully flat map*

$$F_*^{n+1} : W \rightarrow F_*^{n+1} W$$

given by the $(n+1)$ -fold Frobenius.

Lemma 4.22. *The $\mathbb{E}_3$ -MU-algebra map $\mathrm{BP}\langle n \rangle \rightarrow \mathrm{BP}\langle -1 \rangle \cong \mathbb{F}_p$ of [HW22] induces the $\mathbb{F}_p$ -Hopf subalgebra*

$$\mathrm{THH}_*(\mathrm{BP}\langle n \rangle / \mathrm{BP}) / (p, v_1, \dots, v_n) \cong \mathcal{O}_{F_*^{n+1} W}$$

structure of

$$\mathrm{Spec}(\mathrm{THH}_*(\mathbb{F}_p / \mathrm{BP})) \cong \mathcal{O}_W$$

that is equivalent to the faithfully flat graded affine group scheme map

$$F_*^{n+1} : W \rightarrow F_*^{n+1} W.$$

over $\mathrm{Spec}(\mathbb{F}_p)$.

Proof. By Corollary 4.21, we know there is an isomorphism of graded affine group schemes between

$$\mathrm{Spec}(\mathrm{THH}_*(\mathrm{BP}\langle n \rangle / \mathrm{BP}) / (p, v_1, \dots, v_n))$$

and $F_*^{n+1} W$. Let f denote the subalgebra map. We have that

$$f(\sigma^2 v_i) \equiv \sigma^2 v_{i+n} \pmod{\text{units } \mathbb{F}_p^\times}$$

for all $i \geq 0$ by primitiveness and degree considerations. $\square$

We prove Theorem 4.2 part (ii) whose statement we now recall. We have

$$\mathrm{BP}\langle n \rangle_{p=\dots=v_n=0}^{\mathrm{conj}} = [F_*^{m+1} W_{\mathbb{Z}_p} / F_* W_{\mathbb{Z}_p}] / \mathbb{G}_m = [\mathbb{G}_a / W[F^n]] \mathbb{G}_m$$

as affine group schemes over $\mathrm{Spec}(\mathbb{Z}_p[v_1, \dots, v_n]) / \mathbb{G}_m$ where the quotient is associated to $pF^n : F_* W_{\mathbb{F}_p} \rightarrow F_*^{n+1} W_{\mathbb{F}_p}$.

Proof of Theorem B part (ii). By Lemma 4.22 we have that $\mathrm{BP}\langle n \rangle_{p=\dots=v_n=0}^{\mathrm{conj}}$ is the graded quotient stack associated to the graded affine groupoid

$$F_*^{n+1} W \xleftarrow[0 \times pF^n]{\mathrm{id}_{F_*^{n+1} W} \times 0} F_*^{n+1} W \times_{\mathbb{F}_*} W$$

By Lemma 4.19 we have an equivalence

$$[\mathbb{G}_a / W[F^n]] \simeq [F_*^n W / F_* W].$$

of quotient stacks. $\square$

Corollary 4.23 (Corollary 1.4). *We have*

$$\mathrm{Spec}(\mathrm{THH}_*(\mathrm{BP}\langle n \rangle)/(p, \dots, v_n)) \cong \mathrm{BP}\langle n \rangle_{p=v_1=\dots=v_n=0}^{\mathrm{conj}} \cong [F_*^{n+1}W/F_*W]$$

as graded affine group schemes over $\mathrm{Spec}(\mathbb{F}_p)$ and that

$$\bigoplus_{t \in \mathbb{Z}} R\Gamma(\mathrm{BP}_{p=v_1=\dots=v_n=0}^{\mathrm{conj}}, \mathcal{O}\{t\})[2t] \cong \mathbb{F}_p[\mu_{n+1}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_1, \dots, \lambda_{n+1}),$$

with $|\mu_{n+1}| = 2p^{n+1}$ and $|\lambda_i| = 2p^i - 1$, as graded $\mathbb{F}_p$ -algebras. This is a “stacky” version of the calculation

$$\pi_*(\mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathrm{BP}\langle n \rangle))/(p, v_1, \dots, v_n) \cong \mathbb{F}_p[\mu_{n+1}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_1, \dots, \lambda_{n+1})$$

of [Ang26].

Proof. Consider the isomorphism of graded schemes

$$\mathbb{A}^1[2p^{n+1}] \times F_*^{n+1}W \rightarrow F_*^nW$$

over $\mathrm{Spec}(\mathbb{F}_p)$ induced by taking Teichmüller representative and Verschiebung. This gives

$$[F_*^m/F_*W]/G_m \cong \mathbb{A}^1[2p^{n+1}] \times BW[F^{m+1}]$$

Since $R\Gamma(\mathbb{A}^1, \mathcal{O}2p^{n+1}) \cong \mathbb{F}_p[t]$, it remains to argue

$$\bigoplus_{t \in \mathbb{Z}} R\Gamma(BW[F^m], \mathcal{O}\{t\})[2t] \cong \Lambda_{\mathbb{F}_p}(\lambda_1, \dots, \lambda_{n+1}).$$

The Cartier dual $W[F^n]^\vee$ of $W[F^n]$ is $\widehat{W}_n$, the completion of the n -truncated Witt vector group scheme $W_n := W/V^m$ at the origin [Dev23, Proposition C.1], therefore we will study $R\Gamma(BW[F^m], \mathcal{O}\{t\})$ in terms of quasicoherent sheaves on $\widehat{W}_n$. In fact by [Dev23, Proposition C.4], there exists a symmetric monoidal (with respect to the convolution tensor product on $\mathrm{QCoh}(W_n)$) fully faithful functor

$$\Phi : \mathrm{QCoh}(BW[F^n]) \hookrightarrow \mathrm{QCoh}(\widehat{W}_n)$$

so that under the composition

$$\mathrm{QCoh}(BW[F^n]) \xrightarrow{\Phi} \mathrm{QCoh}(\widehat{W}_n) \rightarrow D(\mathcal{O}_{BW[F^n]})$$

we have $\Phi(\mathcal{O}_{BW[F^n]}) = \mathbb{F}_p$. Thus

$$\begin{aligned} R\Gamma(BW[F^n], \mathcal{O}) &\cong R\mathrm{Hom}_{\mathrm{QCoh}(BW[F^n])}(\mathcal{O}, \mathcal{O}) \\ &\cong R\mathrm{Hom}_{\mathrm{QCoh}(BW[F^n])}(\Phi(\mathcal{O}), \Phi(\mathcal{O})) \\ &= \mathrm{Hom}_{D(\mathcal{O}_{BW[F^n]})}(\mathbb{F}_p, \mathbb{F}_p). \end{aligned}$$

Finally one concludes

$$\bigoplus_{t \in \mathbb{Z}} R\Gamma(BW[F^m], \mathcal{O}\{t\})[2t] \cong \Lambda_{\mathbb{F}_p}(\lambda_1, \dots, \lambda_{n+1}).$$

by, for example, resolving $\mathbb{F}_p$ in $D(\mathcal{O}_{BW[F^n]})$ by the Koszul resolution. $\square$

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